

A RELATIONSHIP BETWEEN PASCAL'S TRIANGLE AND FERMAT'S NUMBERS

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There are many relations known among the entries of Pascal's triangle. In [1], Hoggatt discusses the relation between the Fibonacci numbers and Pascal's triangle. He also gives several references to other related works.

Here, we propose to show a relation between the triangle and the Fermat numbers $f_i = 2^{2^i} + 1$ for $i = 0, 1, 2, \dots$. Let $c(n, j)$ be Pascal's triangle, where n represents the row index and j the column index, both indices starting at zero. Let $a[n]$ be the sequence of numbers constructed from Pascal's triangle as follows: construct a new Pascal's triangle by taking the residue of $c(n, j)$ modulo base 2, then, consider each horizontal row of the new triangle as a whole number which is written in binary arithmetic. In symbols, let

$$(1) \quad a[n] = \sum_{j=0}^n c^*(n, j) 2^j \quad n = 0, 1, 2, \dots,$$

where $c^*(n, j)$ is the residue modulo base 2 of $c(n, j)$. The first few terms of this sequence are 1, 3, 5, 15, 17, 51, 85, 255, 257, 771, etc., starting with $a[0]$.

Proposition . The sequence of numbers

$$a[n] = \sum_{j=0}^n c^*(n, j) 2^j \quad n = 0, 1, 2, \dots,$$

constructed from Pascal's triangle, is equal to the sequence of numbers

$$b[n] = (f_k)^{\alpha_0} (f_{k-1})^{\alpha_1} \dots (3)^{\alpha_k} \quad n = 0, 1, 2, \dots,$$

where $n = a_0 a_1 a_2 \dots a_k$ in binary number expansion, and f_i are the Fermat numbers.

Proof. The proof is by induction. For the purpose of starting the induction, let us verify the relation for $a[0]$ through $a[8]$ by means of the following table:

n	n (binary)	$a[n]$ (binary)	$a[n]$ (decimal)	$b[n]$ (Fermat form)
0	000	1	1	1·1·1
1	001	11	3	1·1·3
2	010	101	5	1·5·1
3	011	1111	15	1·5·3
4	100	10001	17	17·1·1
5	101	110011	51	17·1·3
6	110	1010101	85	17·5·1
7	111	11111111	255	17·5·3
8	1000	10000001	257	257·1·1·1

To complete the induction proof, we assume the theorem is true for $n \leq 2^k$, and prove the theorem for the range $2^k < n \leq 2^{k+1}$. We are performing induction on k , and note that the table proves the induction hypothesis for $k = 2$ and 3. If n is in the range $2^k \leq n < 2^{k+1}$, then it has a binary expansion of the form $1a_1 a_2 \dots a_k$. Next, we observe a pattern forming in the binary construction of a_n between the levels 2^k and 2^{k+1} . For example, the above table shows the pattern above $n = 4$ being repeated, in duplicate, side by side, down to level

$n = 7$, but changing at $n = 8$. The reason that this pattern is formed is that Pascal's triangle can be constructed by addition (sums must be reduced modulo 2) with the well known formula

$$c(n-1, r-1) + c(n-1, r) = c(n, r).$$

We will now describe relationship of the numbers below level 2^k to those above 2^k . Since f_k is equal to one plus the number represented by 1 followed by 2^k zeros, we can form $a[2^k + j]$, for $j = 1, 2, \dots, 2^{k-1}$, by multiplying $a[j]$ by f_k . This multiplication has the effect of repeating the pattern above level 2^k , side by side, down to level $2^{k+1} - 1$, which will then consist of 2^{k+1} "ones." If we now construct $a[2^{k+1}]$ using the addition method, we see that it will consist of one plus the number represented by 1 followed by 2^{k+1} zeros. Thus, we have the two relations

$$a[2^k + j] = a[j]f_k \quad \text{for} \quad j = 1, 2, 3, \dots, 2^{k-1}$$

and

$$a[2^{k+1}] = f_{k+1}.$$

If we apply the induction hypothesis to $a[j]$ for $j < 2^k$, then

$$a[n] = (f_k)^1 (f_{k-1})^{\alpha_1} \dots (3)^{\alpha_k} \quad n < 2^{k+1},$$

where

$$n = 1a_1 \dots a_k, \quad \text{and} \quad a[2^{k+1}] = f_{k+1}.$$

This completes the proof.

REMARK. The same proof easily covers the more general case where Pascal's triangle is computed modulo base ϱ . The resulting sequence is then compared to the Fermat numbers to the base ϱ .

REFERENCE

1. V. E. Hoggatt, Jr., "Generalized Fibonacci Numbers in Pascal's Pyramid," *The Fibonacci Quarterly*, Vol. 10, No. 3 (Oct. 1972), pp. 271-276.

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