THE PERIODIC GENERATING SEQUENCE

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Given an integer sequence $S = \{a_1, a_2, \cdots\}$, $a_i > 0$. Form a new sequence $\{r_n\}$ by first choosing two integers r_{-1} and r_0, then setting

$$r_m = r_{m-1}a_m + r_{m-2}, \quad a_m \in S.$$

We call S a *Generating Sequence.*

Notice that for each $r_k \in \{r_n\}$, we can reduce r_k to $r_k = A(k)r_0 + B(k)r_{-1}$, where $A(k)$ and $B(k)$ are integers. Hence $\{r_0, r_{-1}\}$ can be viewed as a "basis" for $\{r_n\}$. Then,

$$r_{-1} = A(-1)r_0 + B(-1)r_{-1} \Rightarrow A(-1) = 0, \qquad B(-1) = 1,$$

$$r_0 = A(0)r_0 + B(0)r_{-1} \Rightarrow A(0) = 1, \qquad B(0) = 0.$$

Theorem 1. Suppose two sequences $\{r'_n\}$ and $\{r''_n\}$ are generated from the same sequence with different choices of r'_{-1}, r'_0 and r''_{-1}, r''_0, then

$$\begin{vmatrix} r'_{k-1} & r'_k \\ r''_{k-1} & r''_k \end{vmatrix} = (-1)^k \begin{vmatrix} r'_{-1} & r'_0 \\ r''_{-1} & r''_0 \end{vmatrix}.$$

Proof. By induction.

<u>Notation</u>: Let

$$L = \begin{bmatrix} A(k) & B(k) \\ A(k-1) & B(k-1) \end{bmatrix}.$$

Notice that

$$\begin{bmatrix} r_k \\ r_{k-1} \end{bmatrix} = L \begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix}$$

Lemma. $\det(L) = (-1)^k$.

Proof.

$$\begin{vmatrix} r'_{k-1} & r'_k \\ r''_{k-1} & r''_k \end{vmatrix} = \begin{vmatrix} A(k-1)r'_0 + B(k-1)r'_{-1} & A(k)r'_0 + B(k)r'_{-1} \\ A(k-1)r''_0 + B(k-1)r''_{-1} & A(k)r''_0 + B(k)r''_{-1} \end{vmatrix}$$

$$= \{A(k)B(k-1) - A(k-1)B(k)\} \begin{vmatrix} r'_{-1} & r'_0 \\ r''_{-1} & r''_0 \end{vmatrix}$$

$$= \det(L) \begin{vmatrix} r'_{-1} & r'_0 \\ r''_{-1} & r''_0 \end{vmatrix}$$

$$\Rightarrow \det(L) = (-1)^k.$$

Theorem 2. Let

$$S = \{a_1, a_2, \cdots\}$$

be the generating sequence for $\{r_n\}$, then

$$A(m) = A(m-1)a_m + A(m-2)$$

$$B(m) = B(m-1)a_m + B(m-2), \quad a_m \in S.$$

Proof. We have

$$r_m = r_{m-1}a_m + r_{m-2} \Rightarrow [A(m)B(m)]\begin{bmatrix} r_0 \\ r_{-1}\end{bmatrix} = [A(m-1)B(m-1)]\begin{bmatrix} r_0 \\ r_{-1}\end{bmatrix}a_m$$
$$+ [A(m-2)B(m-2)]\begin{bmatrix} r_0 \\ r_{-1}\end{bmatrix}$$
$$\Rightarrow [A(m)B(m)] = [A(m-1)a_m + A(m-2)B(m-1)a_m + B(m-2)].$$

Remark: The above theorem shows that $\{A(n)\}$ and $\{B(n)\}$ are also sequences generated by S. Recall that

$$A(-1) = 0, \qquad A(0) = 1; \qquad B(-1) = 1, \qquad B(0) = 0.$$

We shall now investigate what happens when the generating sequence is an infinite periodic sequence

$$P = \{\overline{a_1, \cdots, a_k}\}.$$

We will let k be the period of P for the rest of our work.

Theorem 3. If $\{r_n\}$ is generated from P, then

$$[A(nk+u)B(nk+u)] = [A(u)B(u)]L^n.$$

Proof. Recall

$$L = \begin{bmatrix} A(k) & B(k) \\ A(k-1) & B(k-1)\end{bmatrix} \quad \text{and} \quad \begin{bmatrix} r_k \\ r_{k-1}\end{bmatrix} = L\begin{bmatrix} r_0 \\ r_{-1}\end{bmatrix}.$$

Then

$$r_u = [A(u)B(u)]\begin{bmatrix} r_0 \\ r_{-1}\end{bmatrix}$$

$$r_{k+u} = [A(u)B(u)]\begin{bmatrix} r_k \\ r_{k-1}\end{bmatrix} = [A(u)B(u)]L\begin{bmatrix} r_0 \\ r_{-1}\end{bmatrix}$$

$$r_{2k+u} = [A(u)B(u)]\begin{bmatrix} r_{2k} \\ r_{2k-1}\end{bmatrix} = [A(u)B(u)]L\begin{bmatrix} r_k \\ r_{k-1}\end{bmatrix}$$

$$= [A(u)B(u)]L^2\begin{bmatrix} r_0 \\ r_{-1}\end{bmatrix}.$$

It is easy to see that

$$r_{nk+u} = [A(u)B(u)]L^n\begin{bmatrix} r_0 \\ r_{-1}\end{bmatrix} \Rightarrow [A(nk+u)B(nk+u)]\begin{bmatrix} r_0 \\ r_{-1}\end{bmatrix} = [A(u)B(u)]L^n\begin{bmatrix} r_0 \\ r_{-1}\end{bmatrix}$$

$$\Rightarrow [A(nk+u)B(nk+u)] = [A(u)B(u)]L^n.$$

Corollary.

$$\begin{vmatrix} A(nk+u) & B(nk+u) \\ A(nk+v) & B(nk+v)\end{vmatrix} = (-1)^{nk}\begin{vmatrix} A(u) & B(u) \\ A(v) & B(v)\end{vmatrix}$$

Proof. By Theorem 3, we get

$$\begin{bmatrix} A(nk+u) & B(nk+u) \\ A(nk+v) & B(nk+v)\end{bmatrix} = \begin{bmatrix} A(u) & B(u) \\ A(v) & B(v)\end{bmatrix}L^n \Rightarrow \begin{vmatrix} A(nk+u) & B(nk+u) \\ A(nk+v) & B(nk+v)\end{vmatrix} = \begin{vmatrix} A(u) & B(u) \\ A(v) & B(v)\end{vmatrix}\det(L^n).$$

Theorem 4. If a sequence $\{r_n\}$ is generated from an infinite periodic sequence P with period k, then

$$r_{n+2k} - C(k)r_{n+k} + (-1)^k r_n = 0,$$

where $C(k)$ is a positive integer independent of the choice of r_{-1} and r_0.

Proof. Consider

$$r_{n+2k} + xr_{n+k} + yr_n = 0.$$

Assume the theorem is true except for the existence of x and y. We have

$$r_{n+2k}+xr_{n+k}+yr_n = 0 \Rightarrow \{[A(n+2k)B(n+2k)]+x[A(n+k)B(n+k)]+y[A(n)B(n)]\}\begin{bmatrix} r_0 \\ r_{-1}\end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} A(n+2k)+xA(n+k)+yA(n) = 0 \\ B(n+2k)+xB(n+k)+yB(n) = 0 \end{cases}.$$

These are solvable iff

$$D = \begin{vmatrix} A(n+k) & B(n+k) \\ A(n) & B(n) \end{vmatrix} \neq 0.$$

Then by Theorem 3,

$$[A(n+k)B(n+k)] = [A(n)B(n)]L = [A(n)A(k)+A(k-1)B(n)A(n)B(k)+B(n)B(k-1)]$$

$$\Rightarrow D = \begin{vmatrix} A(n+k) & B(n+k) \\ A(n) & B(n) \end{vmatrix}$$

$$= A(n)A(k)B(n)+A(k-1)B(n)^2-A(n)^2B(k)-A(n)B(n)B(k-1).$$

The only possibilities for making D vanish are either $n = k-1$ or $n = k$.

When $n = k-1$,

$$D = A(k)A(k-1)B(k-1)-A(k-1)^2B(k) = A(k-1)\det(L) \neq 0.$$

When $n = k$,

$$D = A(k-1)B(k)^2-A(k)B(k)B(k-1) = -B(k)\det(L) \neq 0.$$

Hence x and y exist. Then let $n = 0$, we have

$$A(2k)+xA(k)+yA(0) = 0, \qquad B(2k)+xB(k)+yB(0) = 0.$$

Since $A(0) = 1$, $B(0) = 0$, we get

$$x = -B(2k)/B(k), \qquad y = A(k)[B(2k)/B(k)]-A(2k).$$

By Theorem 3, we obtain

$$[A(2k)B(2k)] = [A(0)B(0)]L^2 = [1\ 0]L^2 = [A(k)^2+A(k-1)B(k)A(k)B(k)+B(k)B(k-1)].$$

Thus

$$x = -B(2k)/B(k) = -(A(k)+B(k-1)) \Rightarrow C(k) = A(k)+B(k-1)$$

$$y = A(k)[A(k)+B(k-1)]-[A(k)^2+A(k-1)B(k)]$$

$$= A(k)B(k-1)-A(k-1)B(k) = \det(L) = (-1)^k.$$

<u>Remark</u>. Since $\{A(n)\}$ and $\{B(n)\}$ are also generated from P, then

$$A(n+2k)-C(k)A(n+k)+(-1)^kA(n) = 0 \quad \text{and} \quad B(n+2k)-C(k)B(n+k)+(-1)^kB(n) = 0.$$

By Theorem 3, this leads us to

$$[A(n)B(n)]\{L^2-C(k)L+(-1)^kI\} = 0 \Rightarrow L^2-C(k)L+\det(L)I = 0,$$

I is the identity matrix.

What happens when $P = \{\bar{a}\}$ since k can be chosen as large as one desires?

Theorem 5. Suppose $\{r_n\}$ is generated from $P = \{\bar{a}\}$ such that

$$r_{n+2k}-C(k)r_{n+k}+(-1)^kr_n = 0.$$

Then $\{C(n)\}$ is also a sequence generated from P with $C(0) = 2$, $C(-1) = -a$.

Proof. Recall $C(k) = A(k)+B(k-1)$. Then

$$C(k)-C(k-1)a-C(k-2) = \{A(k)-A(k-1)a-A(k-2)\}-\{B(k-1)-B(k-2)a-B(k-3)\}$$

$$= 0 \Rightarrow C(k) = C(k-1)a+C(k-2).$$

Also,

$$C(0) = A(0)+B(-1) = 2, \qquad C(1) = A(1)+B(0) = a.$$

But then

$$C(1) = C(0)a+C(-1) \Rightarrow C(-1) = -a.$$

Remark. Since $\{C(n)\}$ is generated from $P = \{\bar{a}\}$, there exists another sequence $\{C'(n)\}$ such that

$$C(n+2k) - C'(k)C(n+k) + (-1)^k C(n) = 0.$$

Notice that $\{C'(n)\} = \{C(n)\}$. For example, when $P = \{\bar{1}\}$, then

$$\{A(n)\} = \{f_{n+1}\}$$

and $\{B(n)\} = \{f_n\}$, $C(n) = f_{n+1} + f_{n-1}$, $\{f_n\}$ is the Fibonacci sequence. Remember

$$A(n+2k) - C(k)A(n+k) + (-1)^k A(n) = 0 \Rightarrow f_{n+2k+1} - (f_{k+1} + f_{k-1})f_{n+k+1} + (-1)^k f_{n+1} = 0$$

and

$$B(n+2k) - C(k)B(n+k) + (-1)^k B(n) = 0 \Rightarrow f_{n+2k} - (f_{k+1} + f_{k-1})f_{n+k} + (-1)^k f_n = 0.$$

Also from Theorem 5 and the last remark,

$$C(n+2k) - C'(k)C(n+k) + (-1)^k C(n) = 0 \Rightarrow \{f_{n+2k+1} + f_{n+2k-1}\} - (f_{k+1} + f_{k-1})\{f_{n+k+1} + f_{n+k-1}\}$$
$$+ (-1)^k\{f_{n+1} + f_{n-1}\} = 0.$$

Theorem 6. Suppose $\{r_n\}$ is generated from $P = \{\bar{a}\}$, then there exist x and y such that $u \geqslant s > t \geqslant 0$,

$$r_{n+u} + xr_{n+s} + yr_{n+t} = 0,$$

x and y rational.

Proof. Think of n as k since the periodicity can vary.

Then follow the proof for Theorem 4. Carrying out the proof, we also find that

$$x = -\frac{\begin{vmatrix} A(u) & B(u) \\ A(t) & B(t) \end{vmatrix}}{\begin{vmatrix} A(s) & B(s) \\ A(t) & B(t) \end{vmatrix}}, \qquad y = -\frac{\begin{vmatrix} A(s) & B(s) \\ A(u) & B(u) \end{vmatrix}}{\begin{vmatrix} A(s) & B(s) \\ A(t) & B(t) \end{vmatrix}}.$$

In particular, when $P = \{\bar{1}\}$, we get

$$f_{n+u} - \frac{\begin{vmatrix} f_{u+1} & f_u \\ f_{t+1} & f_t \end{vmatrix}}{\begin{vmatrix} f_{s+1} & f_s \\ f_{t+1} & f_t \end{vmatrix}} f_{n+s} - \frac{\begin{vmatrix} f_{s+1} & f_s \\ f_{u+1} & f_u \end{vmatrix}}{\begin{vmatrix} f_{s+1} & f_s \\ f_{t+1} & f_t \end{vmatrix}} f_{n+t} = 0.$$

For example, when $u = 9$, $s = 6$ and $t = 2$,

$$f_{n+9} - (13/3)f_{n+6} + (2/3)f_{n+2} = 0.$$

We are going to relate some of the above results to Continued Fractions.

A simple purely periodic continued fraction is denoted by $c = [\overline{a_1, \cdots, a_k}]$. If we take $P = \{\overline{a_1, \cdots, a_k}\}$, then immediately we see that $A(n)/B(n)$ is the n^{th} convergent of c. We also know that

$$A(n+2k) - C(k)A(n+k) + (-1)^k A(n) = 0 \quad \text{and} \quad B(n+2k) - C(k)B(n+k) + (-1)^k B(n) = 0.$$

If we regard these as second-order difference equations, then the auxiliary quadratic equation for them is

$$x^2 - C(k)x + (-1)^k = 0$$

and

$$x = \{C(k) \pm \sqrt{C(k)^2 - 4(-1)^k}\}/2, \quad C(k)^2 - 4(-1)^k > 0.$$

Let m_1, m_2 be the distinct zeros such that $|m_1| > |m_2|$, then $A(nk+u) = \alpha_1 m_1^n + \beta_1 m_2^n$,

$$B(nk+u) = \alpha_2 m_1^n + \beta_2 m_2^n, \quad u < k.$$

By choosing the appropriate initial conditions for $\{A(n)\}$ and $\{B(n)\}$, respectively, we can solve for α_1, β_1 and α_2, β_2. One can take $A(u), A(k+u)$ to be the initial conditions for $\{A(n)\}$ and $B(u), B(k+u)$ for $\{B(n)\}$. Then the $(nk+u)^{th}$ convergent of c is given by

$$\frac{A(nk+u)}{B(nk+u)} = \frac{\alpha_1 + \beta_1 (m_2/m_1)^n}{\alpha_2 + \beta_2 (m_2/m_1)^n} .$$

Hence limit of

$$c = \lim_{n\to\infty} \{A(nk+u)/B(nk+u)\} = \alpha_1/\alpha_2 .$$

Notice that α_1 and α_2 are quadratic irrationals. Is the limit unique? Yes, by Theorem 3, we have

$$\begin{vmatrix} A(nk+u) & B(nk+u) \\ A(nk+v) & B(nk+v) \end{vmatrix} = \det(L^n) \begin{vmatrix} A(u) & B(u) \\ A(v) & B(v) \end{vmatrix} = \pm\sigma,$$

σ is a constant. Then

$$\frac{A(nk+u)}{B(nk+u)} - \frac{A(nk+v)}{B(nk+v)} = \frac{\pm\sigma}{B(nk+u)B(nk+v)}$$

As $n \to \infty$,

$$\frac{A(nk+u)}{B(nk+u)} - \frac{A(nk+v)}{B(nk+v)} = 0 .$$

If $c = [a_1, \cdots, a_j, \overline{a_{j+1}, \cdots, a_{j+k}}]$, then take

$$P = \{a_1, \cdots, a_j, \overline{a_{j+1}, \cdots, a_{j+k}}\}$$

as the generating sequence, the limit of c is then given by

$$\lim_{n\to\infty} \frac{A(nk+u+j)}{B(nk+u+j)}, \quad u > 0.$$

Remark. Actually we have proved just now a theorem in continued fractions: A continued fraction c is peridic iff α is a quadratic irrational, for which c is the continued fraction expansion.

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ADDITIVE PARTITIONS II

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Theorem (Hoggatt). The Tribonacci Numbers,

$$1, 2, 4, 7, 13, 24, \cdots, T_{n+3} = T_{n+2} + T_{n+1} + T_n ,$$

with 3 added to the set uniquely *split* the positive integers and each positive integer $n \neq 3$ or $\neq T_m$ is the sum of two elements of A_0 or two elements of A_1. (See "Additive Partitions I," page 166.)

Conjecture. Let A split the positive integers into two sets A_0 and A_1 and be such that $p \notin A \cup \{1,2\}$, and p is representable as the sum of two elements of A_0 or the sum of two elements of A_1. We call such a set *saturated* (that is $A \cup \{1, 2\}$). Krishnaswami Alladi asks: "Does a *saturated* set imply a unique *additive* partition?" My conjecture is that the set $\{1, 2, 3, 4, 8, 13, 24, \cdots\}$ is *saturated* but does not cause a unique split of the positive integers. Here we have added 3 and 8 to the Tribonacci sequence and deleted the 7. Paul Bruckman points out that this fails for 41. EDITOR

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