AMATEUR INTERESTS IN THE FIBONACCI SERIES IV
CALCULATION OF GROUP SIZES OF RESIDUES OF MODULI

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As indicated in a previous paper [1], the statement that the residues of any modulus M of the Fibonacci Series are periodic was investigated. It was found that, in dividing consecutive F_n by M, residues were formed in a Fibonacci-type series until a residue of zero was reached. The succession of residues so formed may be called a *group* and the number of residues in the group, including the terminal zero is the *group size.* (Note: "Group size" is identical numerically to "entry point" found in [2]. Editor.)

If the residue immediately preceding the terminal zero is unity, the next residue will be an exact repetition of the first residues calculated. Therefore, the group ending 1, 0 marks the end of the group and the period. The period may contain 1, 2, or 4 groups. For example, when the modulus $M = 5$, the period contains four groups:

GROUP	RESIDUES
1	1, 1, 2, 3, 0
2	3, 3, 1, 4, 0
3	4, 4, 3, 2, 0
4	2, 2, 4, 1, 0

Note that each group ends in a zero and that the last group (and the period) ends in a 1, 0. Succeeding residues will merely recapitulate the residues in the order shown, starting with the first residue, 1, in the first group.

After calculating the group and period sizes for successive moduli from 2 through 200 (see Table 1), certain regularities were noted, though the table apparently shows nothing of the kind. The group size G_M (but not the period size) of any modulus given in Table 1 can be calculated from the following two rules.

Rule 1. Determine the prime factors of the modulus, such that

$$M = A^{\ell}B^{m}C^{n}\cdots, \tag{1.1}$$

where $A, B, C, \cdots$ are primes and $\ell, m, n, \cdots, \geq 1$. Then the group size G_M for modulus M is the product of the group sizes of moduli equal to these factors, i.e.,

$$G_M = G_{A^{\ell}}\cdot G_{B^m}\cdot G_{C^n}\cdots, \tag{1.2}$$

except that, if any two of the factor group sizes $G_{A^{\ell}}, G_{B^m}, G_{C^n}, \cdots$, contain some common factor D, divide one or the other of the factor group sizes by D so that the quotient obtained is prime relative to the other factor group size in the pair containing that factor D. Continue until all the quotients are prime relative to each other.

Thus:

$$G_{132} = G_{2^2}\cdot G_3\cdot G_{11} = 6\times 4\times 10.$$

The numbers 6, 4, 10 have common factor $D = 2$. Divide 6 by 2, giving quotient 3 which is prime relative to 4. (Note that dividing the 4 by 2 is incorrect because the quotient 2 is not prime relative to 6.) This leaves

$$G_{132} = 3\times 4\times 10.$$

Now, taking the pair 4 and 10, divide 10 by 2, getting 5 which is prime to 4. The final result is

$$G_{132} = 3\times 4\times 5 = 60$$

which will be found to be correct.

Table 1

M	$A^{\ell}B^{m}C^{n}$	$G_{A^{\ell}}G_{B^{m}}G_{C^{n}}$	G_M	M	$A^{\ell}B^{m}C^{n}$	$G_{A^{\ell}}G_{B^{m}}G_{C^{n}}$	G_M	M	$A^{\ell}B^{m}C^{n}$	$G_{A^{\ell}}G_{B^{m}}G_{C^{n}}$	G_M
				34	2×17	3×9	9	67	P	$M+1$	68
2	P	$M+1$	3	35	5×7	5×8	40	68	$2^2\times17$	6×9	18
3	P	$M+1$	4	36	$2^2\times3^2$	6×12	12	69	3×23	4×24	24
4	2^2	2×3	6	37	P	$(M+1)/2$	19	70	$2\times5\times7$	$3\times5\times8$	120
5	P	M	5*	38	2×19	3×18	18	71	P	$M-1$	70
6	2×3	3×4	12	39	3×13	4×7	28	72	$2^3\times3^2$	6×12	12
7	P	$M+1$	8	40	$2^3\times5$	6×5	30	73	P	$(M+1)/2$	37
8	2^3	2×6	6*	41	P	$(M-1)/2$	20	74	2×37	3×19	57
9	3^2	3×4	12	42	$2\times3\times7$	$3\times4\times8$	24	75	3×5^2	4×25	100
10	2×5	3×5	15	43	P	$M+1$	44	76	$2^2\times19$	6×18	18
11	P	$M-1$	10	44	$2^2\times11$	6×10	30	77	7×11	8×10	40
12	$2^2\times3$	6×4	12	45	$3^2\times5$	12×5	60	78	$2\times3\times13$	$3\times4\times7$	84
13	P	$(M+1)/2$	7	46	2×23	3×24	24	79	P	$M-1$	78
14	2×7	3×8	24	47	P	$(M+1)/3$	16	80	$2^4\times5$	12×5	60
15	3×5	4×5	20	48	$2^4\times3$	12×4	12	81	3^4	3×36	108
16	2^4	2×6	12	49	7^2	7×8	56	82	2×41	3×20	60
17	P	$(M+1)/2$	9	50	2×5^2	3×25	75	83	P	$M+1$	84
18	2×3^2	3×12	12	51	3×17	4×9	36	84	$2^2\times3\times7$	$6\times4\times8$	24
19	P	$M-1$	18	52	$2^2\times13$	6×7	42	85	5×17	5×9	45
20	$2^2\times5$	6×5	30	53	P	$(M+1)/2$	27	86	2×43	3×44	132
21	3×7	4×8	8	54	2×3^3	3×36	36	87	3×29	4×14	28
22	2×11	3×10	30	55	5×11	5×10	10	88	$2^3\times11$	6×10	30
23	P	$M+1$	24	56	$2^3\times7$	6×8	24	89	P	$(M-1)/8$	11
24	$2^3\times3$	6×4	12	57	3×19	4×18	36	90	$2\times3^2\times5$	$3\times12\times5$	60
25	5^2	5×5	25	58	2×29	3×14	42	91	7×13	8×7	56
26	2×13	3×7	21	59	P	$M-1$	58	92	$2^2\times23$	6×24	24
27	3^3	3×12	36	60	$2^2\times3\times5$	$6\times4\times5$	60	93	3×31	4×30	60
28	$2^2\times7$	6×8	24	61	P	$(M-1)/4$	15	94	2×47	3×16	48
29	P	$(M-1)/2$	14	62	2×31	3×30	30	95	5×19	5×18	90
30	$2\times3\times5$	$3\times4\times5$	60	63	$3^2\times7$	12×8	24	96	$2^5\times3$	24×4	24
31	P	$M-1$	30	64	2^6	2×24	48	97	P	$(M+1)/2$	49
32	2^5	2×12	24	65	5×13	5×7	35	98	2×7^2	3×56	168
33	3×11	4×10	20	66	$2\times3\times11$	$3\times4\times10$	60	99	$3^2\times11$	12×10	60
								100	$2^2\times5^2$	6×25	150

M = Modulus
P = Prime Number
*Exceptions

Table 1 (Cont'd.)

M	$A^{\ell}B^{m}C^{n}$	$G_{A^{\ell}}G_{B^{m}}G_{C^{n}}$	G_M
101	P	$(M-1)/2$	50
102	$2 \times 3 \times 17$	$3 \times 4 \times 9$	36
103	P	$M+1$	104
104	$2^3 \times 13$	6×7	42
105	$3 \times 5 \times 7$	$4 \times 5 \times 8$	40
106	2×53	3×27	27
107	P	$(M+1)/3$	36
108	$2^2 \times 3^3$	6×36	36
109	P	$(M-1)/4$	27
110	$2 \times 5 \times 11$	$3 \times 5 \times 10$	30
111	3×37	4×19	76
112	$2^4 \times 7$	12×8	24
113	P	$(M+1)/6$	19
114	$2 \times 3 \times 19$	$3 \times 4 \times 18$	36
115	5×23	5×24	120
116	$2^2 \times 29$	6×14	42
117	$3^2 \times 13$	12×7	84
118	2×59	3×58	174
119	7×17	8×9	72
120	$2^3 \times 3 \times 5$	$6 \times 4 \times 5$	60
121	11^2	1110	110
122	2×61	3×15	15
123	3×41	4×20	20
124	$2^2 \times 31$	6×30	30
125	5^3	5×25	125
126	$2 \times 3^2 \times 7$	$3 \times 12 \times 8$	24
127	P	$M+1$	128
128	2^6	2×48	96
129	3×43	4×44	44
130	$3 \times 5 \times 13$	$4 \times 5 \times 7$	105
131	P	$M-1$	130
132	$2^2 \times 3 \times 11$	$6 \times 4 \times 10$	60
133	7×19	8×18	72

M	$A^{\ell}B^{m}C^{n}$	$G_{A^{\ell}}G_{B^{m}}G_{C^{n}}$	G_M
134	2×67	3×68	204
135	$3^3 \times 5$	36×5	180
136	$2^3 \times 17$	6×9	18
137	P	$(M+1)/2$	69
138	$2 \times 3 \times 23$	$3 \times 4 \times 24$	24
139	P	$(M-1)/3$	46
140	$2^2 \times 5 \times 7$	$6 \times 5 \times 8$	120
141	3×47	4×16	16
142	2×71	3×70	210
143	11×13	10×7	70
144	$2^4 \times 3^2$	12×12	12
145	5×29	5×14	70
146	2×73	3×37	111
147	3×7^2	4×56	56
148	$2^2 \times 37$	6×19	114
149	P	$(M-1)/4$	37
150	$2 \times 3 \times 5^2$	$3 \times 4 \times 25$	300
151	P	$(M-1)/3$	50
152	$2^3 \times 19$	6×18	18
153	$3^2 \times 17$	12×9	36
154	$2 \times 7 \times 11$	$3 \times 8 \times 10$	120
155	5×31	5×30	30
156	$2^2 \times 3 \times 13$	$6 \times 4 \times 7$	84
157	P	$(M+1)/2$	79
158	2×79	3×78	78
159	3×53	4×27	108
160	$2^5 \times 5$	24×5	120
161	7×23	8×24	24
162	2×3^4	3×108	108
163	P	$M+1$	164
164	$2^2 \times 41$	6×20	60
165	$3 \times 5 \times 11$	$4 \times 5 \times 10$	20
166	2×83	3×84	84

M	$A^{\ell}B^{m}C^{n}$	$G_{A^{\ell}}G_{B^{m}}G_{C^{n}}$	G_M
167	P	$M+1$	168
168	$2^3 \times 3 \times 7$	$6 \times 4 \times 8$	24
169	13^2	13×7	91
170	$2 \times 5 \times 17$	$3 \times 5 \times 9$	45
171	$3^3 \times 19$	36×18	36
172	$2^2 \times 43$	6×44	132
173	P	$(M+1)/2$	87
174	$2 \times 3 \times 29$	$3 \times 4 \times 14$	84
175	$5^2 \times 7$	25×8	200
176	$2^4 \times 11$	12×10	60
177	3×59	4×58	116
178	2×89	3×11	33
179	P	$M-1$	178
180	$2^2 \times 3^2 \times 5$	$6 \times 12 \times 5$	60
181	P	$(M-1)/2$	90
182	$2 \times 7 \times 13$	$3 \times 8 \times 7$	168
183	3×61	4×15	60
184	$2^3 \times 23$	6×24	24
185	5×37	5×19	95
186	$2 \times 3 \times 31$	$3 \times 4 \times 30$	60
187	11×17	10×9	90
188	$2^2 \times 47$	6×16	48
189	$3^3 \times 7$	36×8	72
190	$2 \times 5 \times 19$	$3 \times 5 \times 18$	90
191	P	$M-1$	190
192	$2^6 \times 3$	48×4	48
193	P	$(M+1)/2$	97
194	2×97	3×49	147
195	$3 \times 5 \times 13$	$4 \times 5 \times 7$	140
196	$2^2 \times 7^2$	6×56	168
197	P	$(M+1)/2$	99
198	$2 \times 3^2 \times 11$	$3 \times 12 \times 10$	60
199	P	$(M-1)/9$	22
200	$2^3 \times 5^2$	6×25	150

As a second example of application of Rule 1, calculate

$$G_{126} = G_2 \cdot G_{3^2} \cdot G_7 = 3 \times 12 = 8.$$

The pair 8 and 12 contain $D = 4$. Divide 12 (not the 8) by 4 to get a quotient of 3 which is prime to 8. Notice that the quotient 3 is not prime to the first factor 3. However, the requirement is that the quotient must be prime to the other number in the pair, not to all the other factor group sizes. So there remains

$$G_{126} = 3 \times 8 \times 3.$$

The two 3's taken as a pair contain $D = 3$ and one of them is reduced by division to 1, making

$$G_{126} = 1 \times 8 \times 3 = 24,$$

which will be found to be correct.

Rule 2. Powers. If M contains only one prime factor A^{ℓ}, then $\ell = 1$.

(2.1) (i) If the final digit in M is 3 or 7, $G_M = (M + 1)/a;$
(ii) If the final digit in M is 1 or 9, $G_M = (M - 1)/a;$

where a is some integer, $a \geqslant 1$; and when $\ell > 1$, then

(2.2) $$G_M = AG_{A^{\ell-1}}.$$

At least up to $M = 200$, there are only two exceptions to Rule 2. For $M = 5$, $G_M = M = 5$. Here, (2.1) does not apply, since 5 is not a final digit mentioned. However, since 5 is the only prime whose terminal digit is 5, this exception is easy to bear. It is interesting to note that G_5 is the average of $(M + 1)/a$ and $(M - 1)/2$ if $a = 1$. The second exception is that for $M = 8$, $G_M = 6$. Going by (2.2), G_{2^3} should be equal to $2G_{2^2} = 2 \times 6 = 12$. If rule (1.1) is applied, which Rule 2 specifically forbids, G_8 comes out as $2 \times 3 = 6$. This exception cannot be explained.

While these rules will enable one to calculate group size, one sould not deprive himself of the pleasure of calculating and recording the individual residues as described in [1]. Of particular interest is the examination of corresponding residues in successive groups. Look for equality of corresponding residues or for two residues whose sum is M. These will normally occur at the aG^{th} residue, where G is one of the factor group sizes and a is an integer, $a \geqslant 1$. Thus, for $M = 200$,

$$G_{2^3} G_{5^2} = 6 \times 25,$$

the 75^{th} residue in each of the groups is 50.

	Group 1			Group 2		
	25^{th}	75^{th}	125^{th}	25^{th}	75^{th}	125^{th}
Residue:	25	50	125	125	50	25

Note the mirror image characteristic. This is again shown in the residues which occur in every sixth place of both groups. These residues always add up to $M = 200$ and are arranged symmetrically about the 75^{th} residue already identified as 50. Thus:

Group 1	Group 2	Group 1	Group 2
8	192	50	50
144	56	64	136
184	16	88	112
168	32	120	80
40	160	72	128
152	48	176	24
96	104	96	104
176	24	152	48
72	128	40	160
120	80	168	32
88	112	184	16
64	136	144	56
50	50	8	192

Note that no residue in Group 1 occurs in Group 2 but that corresponding residues in the two groups add up to $M = 200$. Also, these numbers have other unusual characteristics. Add any two and the sum will be some one or the other of the numbers or, if the sum is greater than 200, subtract $M = 200$ and the remainder will be found somewhere in the list. Subtract any two numbers with the same result. Of course, the reader's inspection has already noted that the numbers above the central 50 are arranged as mirror images of those below.

It is interesting to note that mirror-image molecules (stereoisomers) are of the utmost importance in biochemical considerations and in heredity. Since the connection between the Fibonacci Series and certain facts in heredity has long been noted, perhaps further investigation of the self-reproductive nature of the Fibonacci Series and of its tendency to form mirror images would be fruitful.

REFERENCES

1. Joseph Mandelson, "Amateur Interests in the Fibonacci Series III, Moduli and Residues," *The Fibonacci Quarterly,* Vol. 6, No. 4, October, 1968, pp. 275–278.
2. Brother Alfred Brousseau, "Entry Points and Periods for the Fibonacci Sequence," *Fibonacci and Related Number Theoretic Tables,* The Fibonacci Association, 1972.

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Proof. By pairwise association and use of the relationship [3, p. 285],

$$\psi'(x/S) = \sum_{i=0}^{\infty} 1/(i+x/S)^2$$

which is uniformly convergent for $x \geqslant 1$, one establishes

$$\omega(j; k_1, k_2) = \sum_{i=0}^{\infty} \int_j^{j+k_1} dx/(x+iS)^2$$

$$= (1/S)^2 \int_j^{j+k_1} \psi'(x/S)dx$$

which integrates into the statement (3). The integral form of the psi function occurring in (4) is listed in [4, p. 16] and the integral evaluation is a celebrated theorem of Gauss [3, p. 286; 4, p. 18].

Corollary. Formula (4) can be extended to an arbitrary positive rational argument via the identity [4, p. 16],

$$\psi(n+z) = \psi(z) + \sum_{i=0}^{n-1} 1/(z+i).$$

An ω-series with an arbitrary even number of k_i parameters can be grouped into a series of successive cycles of parametric incrementation within which the terms are pairwise associated. This procedure leads to an expression in terms of the biparameter ω-series, and application of Lemma 2 yields an explicit summation formula in terms of the psi function.

Theorem 1.

$$\omega(j; k_1, \cdots, k_{2n}) = \sum_{i=0}^{n-1} \omega(j+s_{2i}; k_{2i+1}, S-k_{2i+1})$$

$$= (1/S) \sum_{i=0}^{2n-1} (-1)^{i+1} \psi((j+s_{2i})/S).$$

[Continued on page 172.]