A MATRIX SEQUENCE ASSOCIATED WITH A CONTINUED FRACTION EXPANSION OF A NUMBER

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INTRODUCTION

In Section 1, we introduce a matrix sequence each of whose terms is $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, denoted by L, or $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, denoted by R. We call such sequences LR-sequences. A one-to-one correspondence is established between the set of LR-sequences and the continued fraction expansions of numbers in the unit interval. In Section 2, a partial ordering of the numbers in the unit interval is given in terms of the LR-sequences and the resulting partially ordered set is a tree, called the Q-tree. A continued fraction expansion of a number is interpreted geometrically as an infinite path in the Q-tree and conversely. In Section 3, we consider a special function, g, defined on the Q-tree. We show that g is continuous and strictly increasing, but that g is not absolutely continuous. The proof that g is not absolutely continuous is a measure theoretic argument that utilizes Khinchin's constant and the Fibonacci sequence.

1. THE LR-SEQUENCE

We denote the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ by L and the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ by R.

Definition. An LR-sequence is a sequence of 2×2 matrices, $M_1, M_2, \cdots, M_i, \cdots$ such that for each i, $M_i = L$ or $M_i = R$.

We shall represent points in the plane by column vectors with two components. The set $C = \left\{ \binom{\alpha}{\beta} \,\middle|\, \text{both } \alpha \text{ and } \beta \text{ are non-negative and at least one of } \alpha \text{ and } \beta \text{ is positive} \right\}$ will be called the positive cone. Our present objective is to associate with each vector in the positive cone an LR-sequence.

Definition. A vector $\binom{\alpha}{\beta} \in C$ is said to accept the LR-sequence $M_1, M_2, \cdots, M_i, \cdots$ if and only if there is a sequence

$$\binom{\gamma_0}{\delta_0}, \binom{\gamma_1}{\delta_1}, \cdots, \binom{\gamma_i}{\delta_i}, \cdots$$

whose terms are vectors in C, such that

$$\binom{\gamma_0}{\delta_0} = \binom{\alpha}{\beta}$$

and for each $i \geqslant 1$, $\binom{\gamma_{i-1}}{\delta_{i-1}} = M_i \binom{\gamma_i}{\delta_i}$.

If $\binom{\alpha}{\beta} \in C$ and $\alpha \leqslant \beta$, then $\binom{\alpha}{\beta} = \binom{\alpha}{\beta - \alpha}$ and $\binom{\alpha}{\beta - \alpha} \in C$.

If $\beta \leqslant \alpha$, then

$$\binom{\alpha}{\beta} = R\binom{\alpha - \beta}{\beta} \qquad \text{and} \qquad \binom{\alpha - \beta}{\beta} \in C.$$

By induction it can be shown that every vector in C accepts at least one LR-sequence. If α is a positive irrational number, then $\binom{\alpha}{1}$ accepts exactly one LR-sequence; if α is a positive rational number, then $\binom{\alpha}{1}$ accepts two LR-sequences.

The expression $R^{a_0}L^{a_1}R^{a_2}\cdots$ will be used to designate the LR-sequence which consists of a_0 R's, followed by a_1 L's, followed by a_2 R's, etc.

We shall follow Khinchin's notation for continued fractions and express the continued fraction expansion of

$$\alpha,\quad \alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2} + \cdots} \qquad \text{as } \alpha = [a_0; a_1, a_2, \cdots].$$

The remainder after n elements in the expansion of α is denoted by $r_n = [a_n; a_{n+1}, a_{n+2}, \cdots]$. All the well known terms and results of continued fractions used in this paper may be found in [1].

Theorem 1. Let $\alpha = [a_0; a_1, a_2, \cdots]$ and let $\binom{\alpha}{1}$ accept the LR-sequence $R^{b_0}L^{b_1}R^{b_2}\cdots$. Then $b_i = a_i$ for all $i \geq 0$ and for

$$k_n = \sum_{i=0}^{n} b_i,\quad \frac{\gamma_{k_n}}{\delta_{k_n}} = r_{n+1}(\alpha)$$

if n is odd and

$$\frac{\gamma_{k_n}}{\delta_{k_n}} = \frac{1}{r_{n+1}(\alpha)}$$

if n is even.

Proof. Since $\binom{\alpha}{1}$ accepts $R^{b_0}L^{b_1}R^{b_2}\cdots$, there exists a sequence $\binom{\gamma_0}{\delta_0}, \binom{\gamma_1}{\delta_1}, \binom{\gamma_2}{\delta_2}, \cdots$, whose terms are vectors in C, such that $\binom{\gamma_0}{\delta_0} = \binom{\alpha}{1}$ and such that if n is even and $k_n \leq k \leq k_{n+1}$, then

$$\binom{\alpha}{1} = R^{b_0}L^{b_1}R^{b_2}\cdots R^{b_n}L^{k-k_n}\binom{\gamma_k}{\delta_k}$$

and if n is odd, then

$$\binom{\alpha}{1} = R^{b_0}L^{b_1}R^{b_2}\cdots L^{b_n}R^{k-k_n}\binom{\gamma_k}{\delta_k}.$$

Since

$$r_n = [a_n; r_{n+1}],\quad r_{n+1} = \frac{1}{r_n - a_n} \quad\text{and}\quad a_n = [r_n].$$

Therefore a_n is the least integer j such that $r_n - j < 1$.

We now use induction on n. For $n = 0$, $r_0 = \alpha$. Since a_0 is the least integer j such that

$$\alpha - j < 1,\quad \binom{\alpha}{1} = R^{a_0}\binom{\gamma_{a_0}}{\delta_{a_0}},$$

where $\gamma_{a_0} = \alpha - a_0$ and $\delta_{a_0} = 1$. Thus

$$b_0 = a_0 \quad\text{and}\quad \frac{\gamma_{k_0}}{\delta_{k_0}} = \frac{\alpha - a_0}{1} = \frac{1}{r_1}.$$

We assume the result for $0 \leq t < n$ and then consider two cases.

CASE 1. Let n be odd. Then

$$\frac{\gamma_{k_n-b_n}}{\delta_{k_n-b_n}} = \frac{\gamma_{k_{n-1}}}{\delta_{k_{n-1}}} = \frac{1}{r_n} < 1$$

and since a_n is the least integer j such that $r_n - j < 1$,

$$\binom{\gamma_{k_n-b_n}}{\delta_{k_n-b_n}} = L^{a_n}\binom{\gamma_{k_n}}{\delta_{k_n}},\quad \text{where}\quad \gamma_{k_n} = \gamma_{k_n-b_n} \quad\text{and}\quad \delta_{k_n} = \delta_{k_n-b_n} - a_n\gamma_{k_n-b_n}.$$

Thus

$$b_n = a_n \quad\text{and}\quad \frac{\gamma_{k_n}}{\delta_{k_n}} = \frac{\gamma_{k_n-b_n}}{\delta_{k_n-b_n} - a_n\gamma_{k_n-b_n}} = \frac{1}{r_n - a_n} = r_{n+1}.$$

CASE 2. Let n be even. Then

$$\frac{\gamma_{k_n-b_n}}{\delta_{k_n-b_n}} = \frac{\gamma_{k_{n-1}}}{\delta_{k_{n-1}}} = r_n$$

and since a_n is the least integer j such that $r_n - j < 1$,

$$\binom{\gamma_{k_n-b_n}}{\delta_{k_n-b_n}} = R^{a_n}\binom{\gamma_{k_n}}{\delta_{k_n}}, \quad \text{where} \quad \gamma_{k_n} = \gamma_{k_n-b_n} - a_n\delta_{k_n-b_n} \quad \text{and} \quad \delta_{k_n} = \delta_{k_n-b_n}.$$

Thus

$$b_n = a_n \quad \text{and} \quad \frac{\gamma_{k_n}}{\delta_{k_n}} = \frac{\gamma_{k_n-b_n} - a_n\delta_{k_n-b_n}}{\delta_{k_n-b_n}} = r_n - a_n = \frac{1}{r_{n+1}}.$$

The preceding theorem can be extended to hold for rational α by modifying the notation as follows:

(i) If $a_n = 1$, express $[0; a_1, a_2, \cdots, a_n]$ as either

$$[0; a_1, a_2, \cdots, a_n, \infty] \quad \text{or} \quad [0; a_1, a_2, \cdots, a_{n-1} + 1, \infty] \quad \text{or}$$

(ii) If $a_n \neq 1$, express $[0; a_1, a_2, \cdots, a_n]$ as either

$$[0; a_1, a_2, \cdots, a_n - 1, \infty] \quad \text{or} \quad [0; a_1, a_2, \cdots, a_n, \infty].$$

When we permit the use of these expressions we shall speak of continued fractions *in the wider sense.* One sees that the method of LR-sequences provides a common form for the continued fraction expansions for both rational and irrational numbers. (The non-uniqueness, however, of the expansion of a rational number still persists.)

Definition. Let $\alpha = [a_0; a_1, a_2, \cdots]$. The k^{th} order convergent of α is

$$\frac{p_k(\alpha)}{q_k(\alpha)} = [a_0; a_1, a_2, \cdots, a_k],$$

where

$$p_{-1}(\alpha) = 1, \quad p_0(\alpha) = 0, \quad q_{-1}(\alpha) = 0, \quad q_0(\alpha) = 1,$$

and for $k \geqslant 1$,

$$p_k(\alpha) = a_k p_{k-1}(\alpha) + p_{k-2}(\alpha) \quad \text{and} \quad q_k(\alpha) = a_k q_{k-1}(\alpha) + q_{k-2}(\alpha).$$

When no confusion will result, we shall omit the reference to α and write p_k, q_k for $p_k(\alpha)$, $q_k(\alpha)$.

An important proposition in the theory of continued fractions is: If

$$\alpha = [a_0; a_1, a_2, \cdots, a_n, r_{n+1}], \quad \text{then} \quad \alpha = \frac{p_{n+1}}{q_{n+1}} = \frac{r_{n+1}p_n + p_{n-1}}{r_{n+1}q_n + q_{n-1}}.$$

We give an analogue of this result in the following theorem and its corrolary.

Theorem 2. If $\alpha = [0; a_1, a_2, \cdots]$, $\binom{\alpha}{1}$ accepts the LR-sequence $M_1, M_2, \cdots$, and

$$k_n = \sum_{i=1}^{n} a_i,$$

then

$$\prod_{i=1}^{k_n} M_i = \begin{cases} \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} & \text{if } n \text{ is even} \\ \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We use induction on n. For $n = 1$,

$$\prod_{i=1}^{k_1} M_i = L^{a_1} = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} = \begin{pmatrix} p_1 & p_0 \\ q_1 & q_0 \end{pmatrix}.$$

We assume the result for $1 \leqslant t < n$ and then consider two cases.

CASE 1. Let n be even.

$$\prod_{i=1}^{k_n} M_i = \left(\prod_{i=1}^{k_{n-1}} M_i\right) R^{a_n} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2}\end{pmatrix}\begin{pmatrix} 1 & a_n \\ 0 & 1\end{pmatrix} = \begin{pmatrix} p_{n-1} & a_np_{n-1}+p_{n-2} \\ q_{n-1} & a_nq_{n-1}+q_{n-2}\end{pmatrix} = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n\end{pmatrix}.$$

CASE 2. Let n be odd.

$$\prod_{i=1}^{k_n} M_i = \left(\prod_{i=1}^{k_{n-1}} M_i\right) L^{a_n} = \begin{pmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ a_n & 1\end{pmatrix} = \begin{pmatrix} p_{n-2}+a_np_{n-1} & p_{n-1} \\ q_{n-2}+a_nq_{n-1} & q_{n-1}\end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1}\end{pmatrix}.$$

Corollary. If $\alpha = [0; a_1, a_2, \cdots], \binom{\alpha}{1}$ accepts the LR-sequence $M_1, M_2, \cdots$, and

$$k_n = \sum_{i=1}^{n} a_i, \quad \text{then} \quad \binom{\alpha}{1} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1}\end{pmatrix}\binom{\gamma_{k_n}}{\delta_{k_n}}, \quad \text{where} \quad \frac{\gamma_{k_n}}{\delta_{k_n}} = r_{n+1}(\alpha).$$

The well known result,

$$p_{n-1}q_n - p_nq_{n-1} = (-1)^n,$$

is an immediate consequence of the above theorem and the fact that det (L) = det (R) = 1.

2. THE Q-TREE

Although $\binom{\alpha}{1}$ accepts two LR-sequences when α is rational, these two sequences coincide up through a certain initial segment.

Definition. Let α be a positive rational number and let $\binom{\alpha}{1}$ accept the LR-sequence $M_1, M_2, \cdots$. We call the initial segment $M_1, M_2, \cdots, M_n$ a head of α if and only if

$$\binom{\alpha}{1} = M_1, M_2, \cdots, M_n \binom{1}{1}.$$

If α is a positive rational number, the head of α exists and is unique. Thus if $M_1, M_2, \cdots, M_n$ is the head of α, then the two LR-sequences accepted by $\binom{\alpha}{\beta}$ are $M_1, M_2, \cdots, M_n, R, L, L, L, \cdots$ and $M_1, M_2, \cdots, M_n, L, R, R, R, \cdots$.

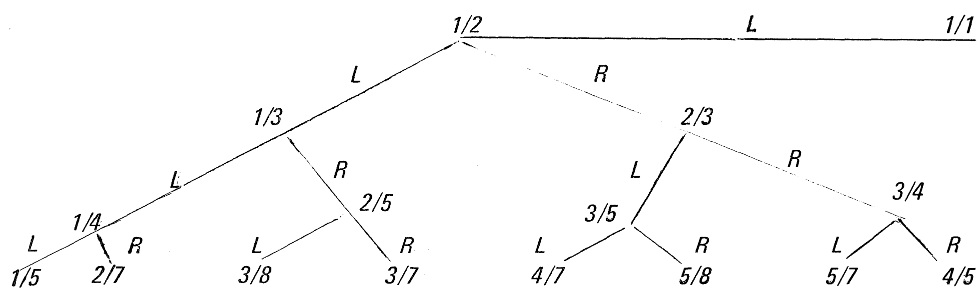
Definition. Let α_1 and α_2 be rational numbers in (0,1]. We say that α_1 is Q-related to α_2 if and only if the head of α_1 is an initial segment of the head of α_2.

The Q relation is a partial ordering of the rational numbers in (0,1], and the resulting partially ordered set is a tree.

Definition. The set of rational numbers in (0,1] partially ordered by Q is called the Q-tree.

We may now interpret the continued fraction expansion of a number (in the wider sense) geometrically as an infinite path in the Q-tree. Conversely, any infinite path in the Q-tree determines an LR-sequence and thus the continued fraction expansion (in the wider sense) for some number.

The following diagram is an indication of the graphical picture of the Q-tree.



(etc.)

Figure 1

3. THE FUNCTION g

Definition. Let $\alpha \in [0,1]$ and let $\binom{\alpha}{1}$ accept the LR-sequence $M_1, M_2, \cdots$. We then define g on the unit interval by

$$g(\alpha) = 2\sum_{j=1}^{\infty} c_j 2^{-j}, \quad \text{where} \quad c_j = \begin{cases} 0 & \text{if } M_j = L \\ 1 & \text{if } M_j = R \end{cases}.$$

It is clear that g is a one-to-one function.

Theorem 3. For $0 \leq \alpha \leq 1$, g is a strictly increasing function.

Proof. Let $0 \leq \alpha < \beta \leq 1$, $\alpha = [0; a_1, a_2, \cdots]$, $\beta = [0; b_1, b_2, \cdots]$ and let t be the least integer n such that $a_n \neq b_n$. Thus $p_k(\alpha) = p_k(\beta)$ and $q_k(\alpha) = q_k(\beta)$ for $0 \leq k < t$.

Now

$$\alpha < \beta \text{ iff } \frac{r_t(\beta)p_{t-1} + p_{t-2}}{r_t(\beta)q_{t-1} + q_{t-2}} - \frac{r_t(\alpha)p_{t-1} + p_{t-2}}{r_t(\alpha)q_{t-1} + q_{t-2}} > 0$$

if and only if

$$r_t(\alpha)(p_{t-2}q_{t-1} - p_{t-1}q_{t-2}) + r_t(\beta)(p_{t-1}q_{t-2} - p_{t-2}q_{t-1}) > 0$$

if and only if

$$(r_t(\alpha) - r_t(\beta))(-1)^{t-1} > 0.$$

Therefore, $r_t(\alpha) > r_t(\beta)$ if and only if t is odd. Since

$$r_t(\alpha) = [a_t; r_{t+1}(\alpha)] \quad \text{and} \quad r_t(\beta) = [b_t; r_{t+1}(\beta)], \quad a_t > b_t$$

if and only if t is odd. We consider two cases determined by the parity of t.

CASE 1. Let t be odd. In this case $a_t > b_t$. If

$$r = \sum_{i=1}^{t} a_i, \quad \text{then} \quad g(\alpha) \leq g\left(\frac{p_{t-1}(\alpha)}{q_{t-1}(\alpha)}\right) + \frac{2}{2^{r-1}}.$$

If

$$s = \sum_{i=1}^{t} b_i, \quad \text{then} \quad g(\beta) \geq g\left(\frac{p_{t-1}(\beta)}{q_{t-1}(\beta)}\right) + \frac{2}{2^s}.$$

Since g is a one-to-one function, $s < r$ and

$$g\left(\frac{p_{t-1}(\alpha)}{q_{t-1}(\alpha)}\right) = g\left(\frac{p_{t-1}(\beta)}{q_{t-1}(\beta)}\right) \quad \text{implies that} \quad g(\alpha) - g(\beta) \leq \frac{2}{2^{r-1}} - \frac{2}{2^s} \leq 0$$

with equality holding if and only if $\alpha = \beta$. Thus $g(\alpha) < g(\beta)$.

CASE 2. Let t be even. In this case $a_t < b_t$ and so $s > r$. Now

$$g(\alpha) \leq g\left(\frac{p_{t-1}(\alpha)}{q_{t-1}(\alpha)}\right) + \frac{2}{2^{r-a_t}}\sum_{i=1}^{a_t}\frac{1}{2^i} + \frac{2}{2^{r+1}} \quad \text{and} \quad g(\beta) \geq g\left(\frac{p_{t-1}(\beta)}{q_{t-1}(\beta)}\right) + \frac{2}{2^{s-b_t}}\sum_{i=1}^{b_t}\frac{1}{2^i}.$$

Since $r - a_t = s - b_t$,

$$g(\alpha) - g(\beta) = \frac{2}{2^{r-a_t}}\left[\sum_{i=1}^{a_t}\frac{1}{2^i} - \sum_{i=1}^{b_t}\frac{1}{2^i}\right] + \frac{2}{2^{r+1}} = -\sum_{i=r+1}^{s}\frac{2}{2^i} + \frac{2}{2^{r+1}} \leq 0$$

with equality holding if and only if $\alpha = \beta$. Thus $g(\alpha) < g(\beta)$.

Corollary. For $\alpha \in [0,1]$, $g'(\alpha)$ exists and is finite almost everywhere.

Theorem 4. For $0 \leq \alpha \leq 1$, g is a continuous function.

Proof. Let $\alpha \in [0,1]$, $\alpha = [0; a_1, a_2, \cdots]$. For any $\epsilon > 0$, choose an n such that

$$\frac{1}{2^{2n}} < \epsilon .$$

Since the even ordered convergents form an increasing sequence converging to α and the odd ordered convergents form a decreasing sequence converging to α, (see [1], p. 6 and p. 9),

$$\frac{p_{2n}}{q_{2n}} < \alpha < \frac{p_{2n+1}}{q_{2n+1}}. \quad \text{Let} \quad \delta = \left|\alpha - \frac{p_{2n+1}}{q_{2n+1}}\right|. \quad \text{Since} \quad \left|\alpha - \frac{p_{2n+1}}{q_{2n+1}}\right| < \left|\alpha - \frac{p_{2n}}{q_{2n}}\right| .$$

If $\beta \in [0,1]$ and $|\alpha - \beta| < \delta$, then either $\dfrac{p_{2n}}{q_{2n}} < \alpha \leqslant \beta < \dfrac{p_{2n+1}}{q_{2n+1}}$ or $\dfrac{p_{2n}}{q_{2n}} < \beta \leqslant \alpha < \dfrac{p_{2n+1}}{q_{2n+1}}$.

Since g is an increasing function,

$$|g(\alpha) - g(\beta)| < \left| g\left(\frac{p_{2n+1}}{q_{2n+1}}\right) - g\left(\frac{p_{2n}}{q_{2n}}\right)\right| = 2\cdot 2^{-\sum_{i=1}^{2n+1} a_i} \leqslant \frac{2}{2^{n+1}} < \epsilon .$$

In the next theorem, we make use of the Fibonacci sequence $<f_n>$, where $f_0 = 1,\ f_1 = 1,$ and $f_n = f_{n-1} + f_{n-2}$.

Theorem 5. The derivative of g at $u = (-1+\sqrt{5})/2$ is infinite.

Proof. The continued fraction expansion of u is $[0; a_1, a_2, \cdots]$, where $a_i = 1$ for all $i \geqslant 1$. Therefore,

$$p_n = p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = q_{n-1} + q_{n-2} .$$

Since $p_{-1} = 1,\ p_0 = 0,\ q_{-1} = 0,\ q_0 = 1,\ p_n = f_n$ and $q_n = f_{n+1}$.

If

$$\frac{p_{2n}}{q_{2n}} < x \leqslant \frac{p_{2n+2}}{q_{2n+2}} < u,$$

then

$$\lim_{x \to u^-} \frac{g(u) - g(x)}{u - x} \geqslant \lim_{n \to \infty} \frac{g(u) - g\left(\frac{p_{2n+2}}{q_{2n+2}}\right)}{u - \frac{p_{2n}}{q_{2n}}} = \lim_{n \to \infty} \frac{\sum_{i=1}^{\infty} \frac{2}{2^{2i}} - \sum_{i=1}^{n+1} \frac{2}{2^{2i}}}{u - \frac{f_{2n}}{f_{2n+1}}},$$

which can be shown equal to (see [2], p. 15)

$$\lim_{n \to \infty} \frac{\sum_{i=n+2}^{\infty} \frac{2}{2^{2i}}}{u\left[1 - \frac{u^{-2n} - u^{2n}}{u^{-2n} + u^{2n+2}}\right]} = \lim_{n \to \infty} \frac{2(1+u^{4n+2})}{3u(u^2 - 1 + 2u^{-4n})}\left(\frac{1}{4u^4}\right)^n .$$

Since

$$\frac{1}{4u^4} = \frac{7+3\sqrt{5}}{8} > 1, \qquad \lim_{x \to u^-} \frac{g(u) - g(x)}{u - x} = \infty .$$

Similarly,

$$\lim_{x \to u^+} \frac{g(u) - g(x)}{u - x} = \infty .$$

We omit the details

Definition. The numbers $\alpha = [a_0; a_1, a_2, \cdots]$ and $\beta = [b_0; b_1, b_2, \cdots]$ are said to be equivalent provided there exists an N such that $a_n = b_n$ for $n \geqslant N$.

Corollary 1. If $\alpha = [a_0; a_1, a_2, \cdots]$ is equivalent to u, then $g'(\alpha) = \infty$.

Proof. Since α is equivalent to u, there exists an N such that $a_n = 1$ for $n \geqslant N$. If

$$\frac{p_{2n}}{q_{2n}} < x \leqslant \frac{p_{2n+2}}{q_{2n+2}} < \alpha < \frac{p_{2n+1}}{q_{2n+1}} ,$$

where $2n \geqslant N$, then

$$\lim_{n \to \alpha^-} \frac{g(\alpha) - g(x)}{\alpha - x} \geqslant \lim_{n \to \infty} \frac{g(\alpha) - g\left(\dfrac{p_{2n+2}}{q_{2n+2}}\right)}{\dfrac{p_{2n+1}}{q_{2n+1}} - \dfrac{p_{2n}}{q_{2n}}} = \lim_{n \to \infty} \sum_{i=n+2}^{\infty} \frac{2}{2^{2i}} (q_{2n} q_{2n+1}).$$

Since $a_n = 1$ for $n \geqslant N$,

$$q_n \geqslant f_n = \frac{u^n - (-u)^{-n}}{\sqrt{5}} \quad .$$

Thus

$$\lim_{n \to \alpha^-} \frac{g(\alpha) - g(x)}{\alpha - x} \geqslant \lim_{n \to \infty} \frac{2}{15 \cdot 4^n} (u^{2n} - u^{-2n})(u^{2n+1} + u^{-2n-1}) = \lim_{n \to \infty} \frac{2}{15} (u^{8n+1} + u^{4n-1} - u^{-1}) \left(\frac{1}{4u^4}\right)^n$$

Since $1/4u^4 = (7 + 3\sqrt{5})/8 > 1$,

$$\lim_{x \to \alpha^-} \frac{g(\alpha) - g(x)}{\alpha - x} = \infty .$$

Similarly

$$\lim_{x \to \alpha^+} \frac{g(\alpha) - g(x)}{\alpha - x} = \infty .$$

Corollary 2. In every subinterval of $[0,1]$ there exists a γ such that $g'(\gamma) = \infty$.

Proof. Let

$$(\alpha, \beta] \subset (0,1], \quad \alpha = [0; a_1, a_2, \cdots] \quad \text{and} \quad \beta = [0; b_1, b_2, \cdots] .$$

We may assume that β is not equivalent to u for if it is, there is nothing to prove.

Let t be the least integer n such that $a_n \neq b_n$. Choosing n such that $2n > t$ and $b_{2n+2} > 1$, we define

$$x = [0; b_1, b_2, \cdots, b_{2n}, \infty], \quad \gamma = [0; b_1, b_2, \cdots, b_{2n+1}, 1, 1, 1, \cdots], \quad \text{and} \quad y = [0; b_1, b_2, \cdots, b_{2n+2}, \infty].$$

Then $\alpha < x < \gamma < y < \beta$ and γ is equivalent to u. Thus the derivative of g at γ is infinite.

The measure used in this next theorem is Lebesgue measure. The measure of a set A is denoted by $m(A)$.

Theorem 6. For almost all $\alpha = [0; a_1, a_2, \cdots] \in (0, 1]$, $g'(\alpha) = 0$.

Proof. Let

$$A = \left\{\alpha \in (0,1] : \lim_{n \to \infty} \left(\prod_{i=1}^{n} a_i \right)^{1/n} = \text{Khinchin's constant} \right\},$$

$$B = \{\alpha \in (0,1] : g'(\alpha) \text{ exists and is finite}\}, \text{ and}$$

$$C = \{\alpha \in (0,1] : a_n > n \log n \text{ for infinitely many values of } n\}.$$

Since (see [1], pp. 93, 94),

$$m(A) = m(B) = m(C) = 1,$$
$$m(A \cap B \cap C) = 1.$$

Let

$$\alpha \in A \cap B \cap C$$

and let $\{x_n\}$ be any sequence converging to α. We define a second sequence $\{y_n\}$ in terms of the partial quotients, p_m/q_m, of α. Let

$$y_n = \left\{ \frac{p_m}{q_m} : m \text{ is the greatest integer such that (i) } |\alpha - x_n| \leqslant \left|a - \frac{p_m}{q_m}\right| \text{ and} \right.$$

$$\left. \text{(ii) } (\alpha - x_n) \text{ and } \left(\alpha - \frac{p_m}{q_m}\right) \text{ have the same sign} \right\}$$

We note that m is an unbounded, non-decreasing function of n and thus m goes to infinity as n does and conversely. Since g is a strictly increasing function and noting that

$$\left|\alpha - \frac{p_{m+2}}{q_{m+2}}\right| < |\alpha - x_n| \quad \text{and that} \quad \left(\alpha - \frac{p_{m+2}}{q_{m+2}}\right)$$

has the same sign as

$$\left(\alpha - \frac{p_m}{q_m}\right),$$

we have

$$\left|\frac{g(\alpha) - g(x_n)}{\alpha - x_n}\right| \leqslant \left|\frac{g(\alpha) - g\left(\frac{p_m}{q_m}\right)}{\alpha - x_n}\right|$$

$$< \left|\frac{g(\alpha) - g\left(\frac{p_m}{q_m}\right)}{\alpha - \frac{p_{m+2}}{q_{m+2}}}\right|$$

$$= \left|g(\alpha) - g\left(\frac{p_m}{q_m}\right)\right| [q_{m+2}(q_{m+2} + q_{m+3}] \qquad \text{[See [1], p. 20.]}$$

$$< \left|g(\alpha) - g\left(\frac{p_m}{q_m}\right)\right| 2q_{m+3}^2$$

$$\leqslant (2\cdot 2^{-k_m})2q_{m+3}^2, \quad \text{where} \quad k_m = \sum_{i=1}^{m} a_i\,.$$

Since Khinchin's constant is < 3,

$$q_m = a_m q_{m-1} + q_{m-2} < 2^m \prod_{i=1}^{m} a_i$$

and $a \in A$, we have that

$$q_{m+3}^2 < \left(2^{m+3} \prod_{i=1}^{m+3} a_i\right)^2 < 2^{2m+6}3^{2m+6}$$

for sufficiently large values of m. Now $\alpha \in C$ implies that $k_m > m \log m$ for infinitely many values of m and thus

$$\left|\frac{g(\alpha) - g(x_n)}{\alpha - x_n}\right| < 2^8\cdot 3^6 \left(\frac{36}{2^{\log m}}\right)^m$$

for infinitely many values of m and n. As n goes to infinity, m goes to infinity and hence given any positive ϵ, the inequality

$$\left|\frac{g(\alpha) - g(x_n)}{\alpha - x_n}\right| < \epsilon$$

will be satisfied for infinitely many values of n. Since $\alpha \in B$, $g'(\alpha)$ exists and therefore $g'(\alpha) = 0$.

Corollary. The function g is not absolutely continuous.

Proof. Since g is not a constant function and for almost all $\alpha \in (0,1]$ $g'(\alpha) = 0$, it follows from a well known theorem that g is not absolutely continuous. (See [3], p. 90.)

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1. A. Ya. Khinchin, *Continued Fractions,* 3rd ed., Chicago: The University of Chicago Press, 1964.
2. N. N. Vorobyov, *The Fibonacci Numbers,* Boston: D. C. Heath and Company, 1963.
3. H. L. Royden, *Real Analysis,* New York: The Macmillan Company, 1963.

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