AN APPLICATION OF THE CHARACTERISTIC OF THE GENERALIZED FIBONACCI SEQUENCE

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1. INTRODUCTION

In [1], Hoggatt and Bicknell discuss the numerator polynomial coefficient arrays associated with the row generating functions for the convolution arrays of the Catalan sequence and related sequences [2], [3]. In this paper, we examine the numerator polynomials and coefficient arrays associated with the row generating functions for the convolution arrays of the generalized Fibonacci sequence $\{H_n\}_{n=1}^{\infty}$ defined recursively by

$$H_1 = 1, \quad H_2 = P, \quad H_n = H_{n-1} + H_{n-2}, \quad n \geq 3, \tag{1}$$

where the characteristic $D = P^2 - P - 1$ is a prime. A partial list of P for which the characteristic is a prime is given in Table 1. A zero indicates that the characteristic is composite, while $P^2 - P - 1$ is given if the characteristic is a prime.

Table 1
Characteristic $P^2 - P - 1$ is Prime, $1 \leq P \leq 179$

	0	1	2	3	4	5	6	7	8	9
0	0	0	0	5	11	19	29	41	0	71
1	89	109	131	0	181	0	239	271	0	0
2	379	419	461	0	0	599	0	701	0	811
3	0	929	991	0	0	0	1259	0	0	1481
4	1559	0	1721	0	0	1979	2069	2161	0	2351
5	0	2549	0	0	2861	2969	3079	3191	0	0
6	3539	3659	0	0	0	4159	4289	4421	0	4691
7	0	4969	0	0	0	0	0	5851	0	0
8	0	0	0	0	6971	0	7309	7481	0	0
9	8009	0	0	0	8741	8929	0	9311	0	0
10	0	10099	10301	0	10711	0	0	0	0	0
11	0	0	0	0	0	13109	13339	0	0	0
12	0	14519	0	0	0	0	15749	16001	0	0
13	0	17029	17291	0	0	18089	0	0	0	19181
14	0	19739	20021	0	0	20879	21169	0	0	22051
15	22349	0	0	0	23561	23869	24179	0	0	25121
16	25439	25759	0	0	26731	27059	0	0	0	0
17	28729	0	29411	0	0	30449	0	31151	0	0

Examining Table 1, we see that $P^2 - P - 1$ is never prime, with the exception of $P = 3$, whenever P is an integer whose units digit is a 3 or an 8. This is so because $P^2 - P - 1 \equiv 0 \pmod 5$ if $P \equiv 3 \pmod 5$. Furthermore, we note that there are some falling diagonals which are all zeros. This occurs whenever $P \equiv -3 \pmod{11}$ or $P \equiv 4 \pmod{11}$.

If P is an integer whose units digit is not congruent to 3 modulo 5, then $P^2 - P - 1 \equiv \pm 1 \pmod 5$ and we see why no prime, in fact no integer, of the form $5k \pm 2$ would occur in Table 1.

There also exist primes of the form $5k \pm 1$ which are not of the form $P^2 - P - 1$. Such primes are 31, 61, 101, 59, 79, and 119. The last observation leads one to question the cardinality of P for which $P^2 - P - 1$ is a prime. The authors believe that there exist an infinite number of values for which the characteristic is a prime. However, the proof escapes discovery at the present time and is not essential for the completion of this paper.

2. A SPECIAL CASE

The convolution array, written in rectangular form, for the sequence $\{H_n\}_{n=1}^{\infty}$, where $P = 3$ is

Convolution Array when $P = 3$

1	1	1	1	1	1	1	1	...
3	6	9	12	15	18	21	24	...
4	17	39	70	110	159	217	284	...
7	38	120	280	545	942	1498	2240	...
11	80	315	905	2120	4311	7910	13430	...
18	158	753	2568	7043	16536	34566	66056	...
...	...	...	...	...	...	...	...	...

The generating function $C_m(x)$ for the m^{th} column of the convolution array is given by

$$C_m(x) = \left[\frac{1+2x}{1-x-x^2}\right]^m \tag{2}$$

and it can be shown that

$$(1+2x)C_{m-1}(x) + (x+x^2)C_m(x) = C_m(x). \tag{3}$$

Using $R_{n,m}$ as the element in the n^{th} row and m^{th} column of the convolution array, we see from (3) that the rule of formation for the convolution array is

$$R_{n,m} = R_{n-1,m} + R_{n-2,m} + R_{n,m-1} + 2R_{n-1,m-1}. \tag{4}$$

Pictorially, this is given by

	a
c	b
d	x

where

$$x = a + b + d + 2c. \tag{5}$$

Letting $R_m(x)$ be the generating function for the m^{th} row of the convolution array and using (4), we have

$$R_1(x) = \frac{1}{1-x} \tag{6}$$

$$R_2(x) = \frac{3}{(1-x)^2} \tag{7}$$

and

$$R_m(x) = \frac{(1+2x)N_{m-1}(x) + (1-x)N_{m-2}(x)}{(1-x)^m} = \frac{N_m(x)}{(1-x)^m}, \quad m \geq 3, \tag{8}$$

where $N_m(x)$ is a polynomial of degree $m - 2$.

The first few numerator polynomials are found to be

$$\begin{aligned}
N_1(x) &= 1\\
N_2(x) &= 3\\
N_3(x) &= 4 + 5x\\
N_4(x) &= 7 + 10x + 10x^2\\
N_5(x) &= 11 + 25x + 25x^2 + 20x^3\\
N_6(x) &= 18 + 50x + 75x^2 + 60x^3 + 40x^4.
\end{aligned}$$

Recording our results by writing the triangle of coefficients for these polynomials, we have

Table 2

Numerator Polynomial $N_m(x)$ Coefficients when $P = 3$

1						
3						
4	5					
7	10	10				
11	25	25	20			
18	50	75	60	40		
29	100	175	205	140	80	
47	190	400	540	530	320	160

It appears as if 5 divides every coefficient of every polynomial $N_m(x)$ except for the constant coefficient.

Using (6), (7), and (8), we see that the constant coefficient of $N_m(x)$ is H_m and it can be shown by induction that

(9) $$H_{n-1}H_{n+1} - H_n^2 = 5(-1)^{n+1}.$$

If 5 divides H_{n-1} then 5 divides H_n and by (1) H_{n-2}. Continuing the process, we have that 5 divides $H_1 = 1$ which is obviously false. Hence, 5 does not divide H_n for any n.

Using (8), we see that the rule of formation for the triangular array of coefficients of the numerator polynomials follows the scheme

d	a
c	b
	x

where

(10) $$x = a + b + 2c - d.$$

By mathematical induction, we see that

(11) $$H_{n+1} = 3F_n + F_{n-1},$$

where F_n is the n^{th} Fibonacci number.

From (10) and (11), we now know that the values in the second column are given by

(12) $$x = a + b + 5F_n.$$

Since 5 divides the first two terms of the second column of Table 2, we conclude using (12), (10), and induction that 5 divides every element of Table 2 which is not in the first column. By induction and (10), it can be shown that the leading coefficient of $N_m(x)$ is given by

(13) $$2^{m-3}\cdot 5, \quad m \geqslant 3.$$

Now in [4], we find

Theorem 1. Eisenstein's Criterion. Let

$$q(x) = \sum_{i=0}^{n} a_i x^i$$

be a polynomial with integer coefficients. If p is a prime such that $a_n \not\equiv 0 \pmod p$, $a_i \equiv 0 \pmod p$ for $i < n$, and $a_0 \not\equiv 0 \pmod{p^2}$ then $q(x)$ is irreducible over the rationals.

In [5], we have

Theorem 2. If the polynomial

$$g(x) = \sum_{i=0}^{n} a_i x^i$$

is irreducible then the polynomial

$$h(x) = \sum_{i=0}^{n} a_{n-i}x^i$$

is irreducible.

Combining all of these results, we have the nice result that $N_m(x)$ is irreducible for all $m \geq 3$. In fact, we shall now show that these results are true for any P such that the characteristic $P^2 - P - 1$ is a prime.

3. THE GENERAL CASE

Throughout the remainder of this paper, we shall assume that P is an integer where $P^2 - P - 1$ is a prime.

By standard techniques, it is easy to show that the generating function for the sequence $\{H_n\}_{n=1}^{\infty}$ is

$$\frac{1+(p-1)x}{1-x-x^2}. \tag{14}$$

By induction, one can show that

$$(1+(p-1)x)\left(\frac{1+x(p-1)}{1-x-x^2}\right)^n + (x+x^2)\left(\frac{1+(p-1)x}{1-x-x^2}\right)^{n+1} = \left(\frac{1+(p-1)x}{1-x-x^2}\right)^{n+1}. \tag{15}$$

Hence, the rule of formation for the convolution array associated with the sequence $\{H_n\}_{n=1}^{\infty}$ is

$$R_{n,m} = R_{n-1,m} + R_{n-2,m} + R_{n,m-1} + (p-1)R_{n-1,m-1}. \tag{16}$$

Since

$$R_1(x) = \frac{1}{1-x} \tag{17}$$

and

$$R_2(x) = \frac{p}{(1-x)^2} \tag{18}$$

we have, by (16) and induction,

$$R_m(x) = \frac{(1+(p-1)x)N_{m-1}(x) + (1-x)N_{m-2}(x)}{(1-x)^m} = \frac{N_m(x)}{(1-x)^m}, \quad m \geq 3. \tag{19}$$

The triangular array for the coefficients of the polynomials $N_m(x)$, with $D = P^2 - P - 1$, is

Table 3

Numerator Polynomial $N_m(x)$ Coefficients when $H_2 = P$

1					
P					
$P+1$	D				
$2P+1$	$2D$	$(P-1)D$			
$3P+2$	$5D$	$(3P-4)D$	$(P-1)^2D$		
$5P+3$	$10D$	$(9P-12)D$	$(4P^2-10P+6)D$	$(P-1)^3D$	
$8P+5$	$20D$	$(22P-31)D$	$(14P^2-36P+23)D$	$(5P^3-18P^2+21P-8)D$	$(P-1)^4D$

By (19), we see that the rule of formation for the triangular array of coefficients of the numerator polynomials $N_m(x)$ follows the scheme

$$\begin{array}{|c|c|} \hline d & a \\ \hline c & b \\ \hline & x \\ \hline \end{array}$$

where

$$x = a + b + (P-1)c - d. \tag{20}$$

By induction, we see that

(21) $$H_{n-1}H_{n+1} - H_n^2 = D(-1)^{n+1}$$
and
(22) $$H_{n+1} = PF_n + F_{n-1},$$

where F_n is the n^{th} Fibonacci number while using (17) through (19) we conclude that the constant term of $N_m(x)$ is H_m.

Following the argument when P was 3 and using (21), we see that D does not divide H_m for any m or that the constant term of $N_m(x)$ is never divisible by D.

By (20) and (22), the elements in the second column of Table 3 are given by

(23) $$x = a + b + F_n D.$$

Since D divides the first two terms of the second column of Table 3, we can conclude by using (23), (20), and induction that D divides every element of Table 3 which is not in the first column. Using (20) and induction, we see that the leading coefficient of $N_m(x)$ is given by

(24) $$(P-1)^{m-3}D, \quad m \geqslant 3.$$

By the preceding remarks, together with Theorems 1 and 2, we conclude that $N_m(x)$ is irreducible for all $m \geqslant 3$, provided D is a prime.

4. CONCLUDING REMARKS

If one adds the rows of Table 2 he obtains the sequence 1, 3, 9, 27, 81, 243, 729, and 2187. Adding the rows of Table 3 we obtain the sequence $1, P, P^2, P^3, P^4, P^5, P^6$, and P^7. This leads us to conjecture that the sum of the coefficients of the numerator polynomial $N_m(x)$ is P^{m-1}.

From (19), we can determine the generating function for the sequence of numerator polynomials $N_m(x)$ and it is

(25) $$\frac{1+(P-1)(1-x)\lambda}{1-(1+(P-1)x)\lambda-(1-x)\lambda^2} = \sum_{m=0}^{\infty} N_{m+1}(x)\lambda^m.$$

Letting $x = 1$, we obtain

(26) $$\frac{1}{1-P\lambda} = \sum_{m=0}^{\infty} (P\lambda)^m = \sum_{m=0}^{\infty} N_{m+1}(1)\lambda^m$$

and our conjecture is proved.

We now examine the generating functions for the columns of Table 3. The generating function for the first column is already given in (14). Using (23), we calculate the generating function for the second column to be

(27) $$C_2(x) = \frac{D}{(1-x-x^2)^2}$$

while when using (20) we see that

(28) $$C_n(x) = \frac{P-1-x}{1-x-x^2}\, C_{n-1}(x), \quad n \geqslant 3.$$

Hence, we have

(29) $$C_1(x) + x^2 C_2(x) \sum_{k=0}^{\infty} \left(\frac{x(P-1)-x^2}{1-x-x^2}\right)^k = \frac{1}{1-xP}.$$

In conclusion, we observe that there are special cases when the characteristic D is not a prime and the polynomials $N_m(x)$ are still irreducible.

In [7], it is shown that

(30) $$D = 5^e P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_n^{\alpha_n}, \quad e = 0 \text{ or } 1,$$

where the P_i are primes of the form $10m \pm 1$.

Assume either $e = 1$ or some $a_i = 1$. Following the argument when P was 3 and using (21), we conclude that neither 5 nor P_i divides the constant term of $N_m(x)$. We have already shown that D divides every nonconstant coefficient of every polynomial $N_m(x)$ so that either 5 or P_i divides every nonconstant coefficient of every polynomial $N_m(x)$.

By Theorems 1 and 2 together with (24), we now know that the polynomials $N_m(x)$ are irreducible whenever 5 or P_i does not divide $P - 1$. However, it is a trivial matter to show that neither 5 nor P_i can divide both $P - 1$ and $P^2 - P - 1 = D$. Hence, $N_m(x)$ is irreducible for all $m \geq 3$ provided $e = 1$ or $a_i = 1$ for some i.

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7. Dmitri Thoro,

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METRIC PAPER TO FALL SHORT OF "GOLDEN MEAN"

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If the Greeks were right that the most pleasing of rectangles were those having their sides in medial section ratio, $\sqrt{5} + 1 : 2$, the classic "Golden Mean," then the world is missing a golden opportunity in standardizing its paper sizes for the anticipated metric conversion.

Metric paper sizes have their dimensions in the ratio $1 : \sqrt{2}$, an ingenious arrangement that permits repeated halvings without altering the ratio. But the 1.414 ratio of length to width falls perceptively short of the "golden" 1.612, as have most paper sizes with which North Americans are familiar. Thus, 8½ × 11 inch typing paper has the ratio 1.294. Popular sizes for photographic paper include 5 × 7 inches (1.400), 8 × 10 inches (1.250), and 11 × 14 inches (1.283). Closest to the Golden Mean, perhaps, was "legal" size typing paper, 8½ × 14 inches (1.647).

With a number of countries, including the United Kingdom, South Africa, Canada, Australia, and New Zealand, making marked strides into "metrication," office typing paper now is being seen that is a little narrower, a little longer, and notably closer to what the Greeks might have chosen.

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