ZERO-ONE SEQUENCES AND STIRLING NUMBERS OF THE FIRST KIND

C. J. PARK
San Diego State University, San Diego, California 92182

This is a dual note to the paper [1]. Let $x_1, x_2, \cdots, x_n$ denote a sequence of zeros and ones of length n. Define a polynomial of degree $(n-m) \geqslant 0$ as follows

$$(1) \qquad a_{m+1,n+1}(d) = \sum (x_1 - d)^{1-x_1}(x_2 - (d+1))^{1-x_2} \cdots (x_n - (d+n-1))^{1-x_n}$$

with

$$a_{1,1}(d) = 1 \quad \text{and} \quad a_{m+1,n+1}(d) = 0, \quad n < m,$$

where the summation is over $x_1, x_2, \cdots, x_n$ such that

$$\sum_{i=1}^{n} x_i = m.$$

Summing over x_n we have the following recurrence relation

$$(2) \qquad a_{m+1,n+1}(d) = -(d+n-1)a_{m+1,n}(d) + a_{m,n}(d),$$

where

$$a_{0,0}(d) = 1 \quad \text{and} \quad a_{0,n}(d) = 0, \quad n > 0.$$

Summing over x_1, we have the following recurrence relation

$$(3) \qquad a_{m+1,n+1}(d) = -d\,a_{m+1,n}(d+1) + a_{m,n}(d+1),$$

where

$$a_{0,0}(d) = 1 \quad \text{and} \quad a_{0,n}(d) = 0, \quad n > 0.$$

The following theorem establishes a relationship between the polynomials defined in (1) and Stirling numbers of the first kind; see Riordan [2, pp. 32–34].

Theorem 1. $a_{m,n}(1)$ defined in (1) are Stirling numbers of the first kind.

Proof. From (1) $a_{1,1}(d) = 1$ and from (2)

$$(4) \qquad a_{m+1,n+1}(1) = -n\,a_{m+1,n}(1) + a_{m,n},$$

which is the recurrence relation for Stirling numbers of the first kind, see Riordan [2, p. 33]. Thus Theorem 1 is proved.

Using (2), (3) and (4) the following Corollary can be shown.

Corollary. (a) $$a_{m+1,n+1}(0) = a_{m,n}(1)$$

(b) $$a_{m+1,n+1}(1) = -a_{m+1,n}(2) + a_{m,n}(2)$$

(c) $$a_{m,n}(2) - a_{m+1,n}(2) = -na_{m+1,n}(1) + a_{m,n}(1).$$

Theorem 2. Let $\beta_{m+1,n+1}(d)$ be a polynomial of degree $(n-m) \geqslant 0$ given by Park [1]. Then

$$(5) \qquad \sum a_{m+1,k+1}(d)\beta_{k+1,n+1}(d) = \delta_{m+1,n+1} \quad \text{with } \delta_{m,n} \text{ the Kronecker delta.}$$

$\delta_{m,n} = 1$, $\delta_{m,n} = 0$, $m \neq n$, and summed over all values of k for which $a_{m+1,k+1}(d)$ and $\beta_{k+1,n+1}(d)$ are nonzero.

Proof. It can be verified that the polynomial defined in (1) has a generating function

(6) $$(t-d)^{(n)} = \sum_{m=0}^{n} t^m a_{m+1,n+1}(d), \quad \text{where} \quad (t-d)^{(n)} = (t-d)(t-d-1)\cdots(t-d-n+1).$$

The generating function of $\beta_{m+1,n+1}(d)$ can be written

(7) $$t^n = \sum_{m=0}^{n} (t-d)^{(m)}\beta_{m+1,n+1}(d).$$

Using (6) and (7), (5) follows. This completes the proof of Theorem 2.

EXAMPLE: For $n = 3$, let

$$A = \begin{bmatrix} a_{1,1}(d) & 0 & 0 & 0 \\ a_{1,2}(d) & a_{2,2}(d) & 0 & 0 \\ a_{1,3}(d) & a_{2,3}(d) & a_{3,3}(d) & 0 \\ a_{1,4}(d) & a_{2,4}(d) & a_{3,4}(d) & a_{4,4}(d) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -d & 1 & 0 & 0 \\ d(d+1) & -(2d+1) & 1 & 0 \\ -d(d+1)(d+2) & (3d^2+6d+2) & -3(d+1) & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} \beta_{1,1}(d) & 0 & 0 & 0 \\ \beta_{1,2}(d) & \beta_{2,2}(d) & 0 & 0 \\ \beta_{1,3}(d) & \beta_{2,3}(d) & \beta_{3,3}(d) & 0 \\ \beta_{1,4}(d) & \beta_{2,4}(d) & \beta_{3,4}(d) & \beta_{4,4}(d) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ d & 1 & 0 & 0 \\ d^2 & (2d+1) & 1 & 0 \\ d^3 & (3d^2+3d+1) & 3(d+1) & 1 \end{bmatrix}$$

Then $A \cdot B = I$.

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REFERENCES

1. C. J. Park, "Zero-One Sequences and Stirling Numbers of the Second Kind," *The Fibonacci Quarterly*, Vol. 15, No. 3 (Oct. 1977), pp.205–206.
2. John Riordan, *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958.

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PROBLEMS

GUY A. R. GUILLOT
Montreal, Quebec, Canada

Proposed by Guy A. R. Guillot, Montreal, Quebec, Canada.

Prove that

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{1}{n^2+n+1} = \sum_{n=1}^{\infty} \tan^{-1} \frac{1}{F_{2n+1}}.$$

Proposed by Guy A. R. Guillot, Montreal, Quebec, Canada.

Show that

(a) $$\sum_{n=1}^{\infty} \frac{1}{n^2 F_{n+2}} > \frac{\pi^2}{12} - \frac{(\log 2)^2}{2} + \frac{1}{48}$$

(b) $$\sum_{n=0}^{\infty} \frac{1}{F_{n+2}} \left(\tan \frac{\pi}{2^{n+2}}\right) > \frac{4}{\pi} + 0.0166.$$

[Continued on p. 257.]