ON POWERS OF THE GOLDEN RATIO*

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The golden ratio $\underline{G}$ is peculiar in that it is the number $\underline{X}$ such that $\underline{X}^2 = \underline{X} + 1$. This characteristic permits deduction of properties of $\underline{G^n}$ not unlike those of Fibonacci numbers $\underline{F}$. Also, interesting relations of $\underline{F}$ numbers are derivable from properties of $\underline{G^n}$. Some of these properties and relations are given below.

First, a given $\underline{n}^{th}$ power of $\underline{G}$ is the sum of G^{n-1} and G^{n-2} for

$$(1) \qquad G^{n-1} + G^{n-2} = G^{n-2}(G+1) = G^n .$$

Furthermore, for $\underline{n}$ a positive integer, $G^n = F_n G + F_{n-1}$ which implies that G^n approaches an integer as $\underline{n}$ increases. For proof, determine that

$$G^1 = 1G + 0$$

$$G^2 = G + 1 = 1G + 1$$

$$G^3 = G(G+1) = 2G + 1$$

and from (1), $G^4 = (1+2)G + (1+1)$, $G^5 = (3+2)G + (2+1)$, etc.

The coefficient of $\underline{G}$ on the right for each successive power of $\underline{G}$ is the sum of the two preceding F_{n-1} and F_{n-2} coefficients, and the number added to the multiple of $\underline{G}$ is the sum of F_{n-2} and F_{n-3}. Hence,

$$G^n = F_n G + F_{n-1} .$$

As $\underline{n}$ increases, $F_n G \to F_{n+1}$, so

$$(2) \qquad G^n \to F_{n+1} + F_{n-1} .$$

Hence, G^n approaches an integer as $\underline{n}$ incraases, and thus approximates all properties of $F_{n+1} + F_{n-1}$.

No restrictions were placed on $\underline{n}$ in (1), so the equation holds for $n \leqslant 0$. For example, given $\underline{n} = 0$,

$$G^{n-1} + G^{n-2} = \frac{1}{G} + \frac{1}{G^2} = \frac{G+1}{G^2} = 1 = G^0 .$$

Hence, sums of reciprocals of F numbers assume F properties as $F_{n+1}/F_n \to G$. Generally, let $F_n G$ represent F_{n+1}, and $F_n G^2$ represent F_{n+2}. Then

$$(3) \qquad \frac{1}{F_{n+1}} + \frac{1}{F_{n+2}} \to \frac{1}{F_n G} + \frac{1}{F_n G^2} = \frac{1}{F_n}\left(\frac{G+1}{G^2}\right) = \frac{1}{F_n}$$

Equation (3) is a special case of a much more general interpretation of (1), for positive or negative fractional exponents may be used. To reveal the general application to $\underline{F}$ numbers, derive from the general equation for F_n,

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} = \frac{G^n - \frac{1}{(-G)^n}}{\sqrt{5}}$$

that $F_n\sqrt{5} \to G^n$ as $\underline{n}$ increases. Hence, for any positive integers $\underline{n}$ and $\underline{m}$,

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$$G^{\frac{n}{m}} = G^{\frac{n}{m}-1} + G^{\frac{n}{m}-2}$$

(4) $$(G^n)^{\frac{1}{m}} = (G^{n-m})^{\frac{1}{m}} + (G^{n-2m})^{\frac{1}{m}}$$

(5) $$F_n^{\frac{1}{m}} \to F_{n-m}^{\frac{1}{m}} + F_{n-2m}^{\frac{1}{m}} \quad .$$

To illustrate Eq. (4), let $n = 1$ and $m = 3$.

$$G^{\frac{1}{3}} = G^{-\frac{1}{3}} + G^{-\frac{5}{3}} \quad .$$

Cubing both sides gives

$$G = G^{-\frac{6}{3}} + 3G^{-\frac{9}{3}} + 3G^{-\frac{12}{3}} + G^{-\frac{15}{3}} = G^{-5}(G^6) = G\,.$$

The proximity of the relation in (5) even for $\underline{n}$ small can be illustrated by letting $\underline{n} = 10$ and $\underline{m} = 2$, or

$$F_{10}^{1/2} \to F_8^{1/2} + F_6^{1/2}$$

$$\sqrt{55} = 7.416 \to \sqrt{21} + \sqrt{8} = 7.411.$$

Equation (4) adapts readily to $-1/m$, for

$$(G^n)^{-\frac{1}{m}} = (G^{n+m})^{-\frac{1}{m}} + (G^{n+2m})^{-\frac{1}{m}}$$

and from (5),

$$F_n^{-\frac{1}{m}} \to F_{n+m}^{-\frac{1}{m}} + F_{n+2m}^{-\frac{1}{m}} \quad .$$

Again, letting $n = 10$ and $m = 2$,

$$F_{10}^{-1/2} = .134839 \qquad \text{and} \qquad F_{12}^{-1/2} + F_{14}^{-1/2} = .134835.$$

An additional insight regarding $\underline{F}$ relations derives from (2) and the fact that $F_n\sqrt{5} \to G^n$, for

$$F_n\sqrt{5} \to G^n \to F_{n+1} + F_{n-1}$$

$$F_n\sqrt{5} \to F_{n+1} + F_{n-1} \quad .$$

Hence, $F_n\sqrt{5}$ approaches an integer as $\underline{n}$ increases.

These relations of $\underline{F}$ and powers of $\underline{G}$, especially those involving negative exponents, permit greater perspective for $\underline{F}$ numbers. For example, Vorob'ev [1] states that the condition $U_n = U_{n-1} + U_{n-2}$ does not define all terms in the $\underline{F}$ sequence because not every term has two preceding it. Specifically, 1,1,2 ··· does not have two terms before 1,1. Such is not true of G^n where $-\infty < n < \infty$. F_n properties approach those of G^n as $n \to \pm\infty$, with maximum discrepancy at $\underline{n} = 0$. $\underline{G}$ is usually viewed as the limit of F_{n+1}/F_n as $\underline{n} \to \infty$; perhaps the more mystical concept of a guiding essence for harmonic variations of F_n is in order. G^n brings F_n to taw. The distortion in F_n relations relative to G^n is never great so long as $\underline{n}$ is a positive or negative integer. And G^n properties surmount even $\underline{n} = 0$.

A last look at G^n will be made in terms of logarithms of $\underline{F}$ numbers to the base $\underline{G}$. Because $F_n \to G^n/\sqrt{5}$,

$$\log_G F_n \to n - \tfrac{1}{2}\log_G 5 = n - 1.6722759\cdots = (n-2) + .3277240\cdots.$$

Therefore,

(8) $$F_n \to G^{n-2}G^{.3277240\cdots}.$$

Hence, $\log_G F_n - \log_G F_{n-1}$ harmonically approaches unity, and rapidly.

REFERENCE

1. N. N. Vorob'ev, *Fibonacci Numbers,* Blaisdell Publishing Co., New York, 1961, p. 5.

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