FIBONACCI SEQUENCE AND EXTREMAL STOCHASTIC MATRICES

R. A. BRUALDI[1]
University of Wisconsin, Madison, Wisconsin 53706
and
J. CSIMA[2]
McMaster University, Hamilton, Ontario, Canada L8S 4K1

ABSTRACT

The purpose of this note is to exhibit an interesting connection between the Fibonacci sequence and a class of three-dimensional extremal plane stochastic matrices.

1. A *three-dimensional matrix* of order n is a real valued function A defined on the set $J_{3,n}$ of points (i,j,k), where $1 \leqslant i,j,\ \ k \leqslant n$. It is customary to say that the value of this function at the point (i,j,k) is an entry of the matrix and to denote it by $a_{i,j,k}$. A *plane* is defined to be a subset of which results when one of i,j,k is held fixed. A plane is called a *row, column, or horizontal plane* according as to whether i, j, or k is held fixed. A matrix A is *plane stochastic* if its entries are non-negative numbers and the sum of the entries in each plane is equal to one. If A and B are plane stochastic matrices of order n and $0 \leqslant \alpha \leqslant 1$, then $\alpha A + (1-\alpha)B$ is also a plane stochastic matrix. Thus the collection of all plane stochastic matrices of order n is a convex set. The extreme points of this convex set are called *extremal plane stochastic matrices.* Jurkat and Ryser [3] have raised the question of determining all the extremal stochastic matrices. This appears to be a very difficult problem. One class of extremal plane stochastic matrices is formed by the permutation matrices (with precisely one non-zero entry in each plane). But unfortunately very little is known about other extremal matrices.

In what follows we construct a class of extremal plane stochastic matrices using Fibonacci numbers.

2. If A is a three-dimensional matrix of order n, then the *pattern* of A is the set of all points (i,j,k) for which $a_{ijk} \neq 0$. Jurkat and Ryser [3] observed that a *plane Stochastic matrix* A *is extremal if and only if there is no plane stochastic matrix other than* A *which has the same pattern as* A.

We are now ready to construct a class of extremal stochastic matrices. Let $S_n \subseteq J_{3,n}$ $(n = 1, 2, \cdots)$ be the pattern defined as follows: The points $(n, n, n-1)$ and $(1, n, n)$ belong to S_n. In addition $(i,j,k) \in S_n$ whenever one of the following holds:

(i) $\quad i = j = k \quad$ for $\quad i = 1, \cdots, n-1;$

(ii) $\quad i = j+1 \quad$ and $\quad k = n, \quad$ for $\quad i = 2, \cdots, n;$

(iii) $\quad i = j-1 = k+1, \quad$ for $\quad i = 2, \cdots, n-1.$

The matrix T_n in Figure 1 is a two-dimensional representation of this pattern. The (i,j)-entry of T_n equals k if and only if $(i,j,k) \in T_n$. Fortunately, T_n is such that $(i,j,k), (i,j,k') \in T_n$ implies $k = k'$.

The (two-dimensional) matrix B_n indicated in Figure 2 represents a three-dimensional matrix A_n of order n. If $(i,j,k) \in S_n$, then $a_{ijk} = b_{ij}$; if $(i,j,k) \notin S_n$, then $a_{ijk} = 0$. The sequence $f_1, f_2, f_3, f_4, \cdots$ is the Fibonacci sequence 1, 1, 2, 3, $\cdots$.

Theorem. The matrix A_n is an extremal plane stochastic matrix of order n.

Proof. We observe that all the indicated entries of B_n are positive so that the pattern of A_n is S_n. In order to verify that A_n is plane stochastic, we compute the plane sums of A_n. First we observe that the row

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$$\begin{bmatrix} 1 & & & & & & & n \\ n & 2 & 1 & & & & & \\ & n & 3 & 2 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & n & n-3 & n-4 & & \\ & & & & n & n-2 & n-3 & \\ & & & & & n & n-1 & n-2 \\ & & & & & & n & n-1 \end{bmatrix}$$

Figure 1

$$B_n = \frac{1}{f_{n+2}-1}\begin{bmatrix} f_{n+2}-f_3 & & & & & & & f_1 \\ f_2 & f_{n+2}-f_4 & f_3-1 & & & & & \\ & f_3 & f_{n+2}-f_5 & f_4-1 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & f_{n-3} & f_{n+2}-f_{n-1} & f_{n-2}-1 & & \\ & & & & f_{n-2} & f_{n+2}-f_n & f_{n-1}-1 & \\ & & & & & f_{n-1} & f_{n+2}-f_{n+1} & f_n-1 \\ & & & & & & f_n & f_{n+1}-1 \end{bmatrix}$$

Figure 2

and column plane sums of A_n are the row and column sums of B_n. It is more convenient to verify that the row and column sums of $C_n = (f_{n+2} - 1)B_n$ are all $f_{n+2} - 1$. The first row sum of C_n equals

$$f_{n+2} - f_3 + f_1 = f_{n+2} - 1;$$

the last row sum of C_n is clearly $f_{n+2} - 1$. The i^{th} row sum of C_n, $2 \leq i \leq n-1$, equals

$$f_i + (f_{n+2} - f_{i+2}) + (f_{i+1} - 1) = f_{n+2} - 1.$$

The first, second, and last column sums of C_n equal $f_{n+2} - 1$. The j^{th} column sum of C_n, $3 \leq j \leq n-1$, equals

$$(f_j - 1) + (f_{n+2} - f_{j+2}) + f_{j+1} = f_{n+2} - 1.$$

Thus far we have verified that the row and column plane sums of A_n are one. Now we compute the horizontal plane sums of A_n. The k^{th} horizontal plane sum of A_n, $1 \leq k \leq n-1$, equals

$$\frac{(f_{n+2} - f_{k+2}) + (f_{k+2} - 1)}{f_{n+2} - 1} = 1.$$

The n^{th} horizontal plane sum equals

$$\frac{f_1 + f_2 + \cdots + f_n}{f_{n+2} - 1} = 1.$$

Thus A_n is a plane stochastic matrix.

To show that A_n is extremal, it suffices to show that there is no other plane stochastic matrix with pattern S_n.

Let E be a plane stochastic matrix with pattern S_n. Let $a > 0$ be the $(1,n,n)$-entry of E. Let $G = [g_{ij}]$ be the two-dimensional representation of E. We claim that G has the form indicated in Figure 3.

$$G_n = \begin{bmatrix} 1-(f_3-1)a & & & & & & f_1 a \\ f_2 a & 1-(f_4-1)a & (f_3-1)a & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & f_{n-1}a & 1-(f_{n+1}-1)a & (f_n-1)a \\ & & & & & f_n a & 1-f_n a \end{bmatrix}$$

Figure 3

We verify this by using step-by-step the fact that the plane sums of E are one. For the first row of G we have $g_{1n} = a = f_1 a$, and from $g_{11} + g_{1n} = 1$ we conclude that

$$g_{11} = 1 - a = 1 - (f_3 - 1)a.$$

For the second row of G we have $g_{21} = 1 - g_{11} = a = f_2 a$. Since the first horizontal plane sum is one,

$$g_{23} = 1 - g_{11} = a = (f_3 - 1)a.$$

Finally, $g_{22} = 1 - (f_4 - 1)a$ can be determined by considering the second row sum of G. Suppose it has been verified that the first i rows of G are as claimed for some i with $2 \leq i \leq n-1$. Considering the i^{th} column sum of G we compute that

$$g_{i+1,i} = 1 - g_{i-1,i} - g_{ii} = 1 - (f_i - 1)a - (1 - (f_{i+2} - 1)a) = f_{i+1}a.$$

Considering the i^{th} horizontal plane sum of E, we compute that

$$g_{i+1,i+2} = 1 - g_{ii} = 1 - (1 - (f_{i+2} - 1)a) = (f_{i+2} - 1)a.$$

Finally, considering the $(i+1)^{st}$ row sum of G, we compute that

$$g_{i+1,i+1} = 1 - g_{i+1,i} - g_{i+1,i+2} = 1 - f_{i+1}a - (f_{i+2} - 1)a = 1 - (f_{i+3} - 1)a.$$

Thus by induction we have verified our claim up to and including the $(n-1)^{st}$ row of G. By considering the $(n-1)^{st}$ column sum and n^{th} row sum of g in turn, we calculate that

$$g_{n,n-1} = 1 - g_{n-2,n-1} - g_{n-1,n-1} = 1 - (f_{n-1} - 1)a - (1 - (f_{n+2} - 1)a) = f_n a,$$

and

$$g_{n,n} = 1 - g_{n,n-1} = 1 - f_n a.$$

Thus our claim is verified.

Now by considering the n^{th} horizontal plane sum of E, we see that a is uniquely determined. Hence E is unique, and thus $E = A_n$. This completes the proof of the theorem.

Constructions for other extremal matrices and additional properties of planar stochastic matrices can be found in [1, 2].

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BELL'S IMPERFECT PERFECT NUMBERS

EDWART T. FRANKEL
Schenectady, New York

A perfect number is one which, like 6 or 28, is the sum of its aliquot parts. Euclid proved that $2^{p-1}(2^p - 1)$ is perfect when $(2^p - 1)$ is a prime; and it has been shown that this formula includes all perfect numbers which are even.[1]

In Eric Temple Bell's fascinating book[2], the seven perfect numbers after 6 are listed as follows:

28, 496, 8128, 130816, 2096128, 33550336, 8589869056.

Checking these numbers by Euclid's formula, I found that

$$2^8(2^9 - 1) = 256 \times 511 = 130816$$

and

$$2^{10}(2^{11} - 1) = 1024 \times 2047 = 2096128.$$

However, $511 = 7 \times 73$; and $2047 = 23 \times 89$.

Inasmuch as 511 and 2047 are not primes, it follows that 130816 and 2096128 are not perfect numbers, and they should not have been included in Bell's list.

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[1] *Encyclopedia Britannica,* Eleventh Edition, Vol. 19, page 863.
[2] *The Last Problem*, Simon and Schuster, New York, 1961, page 12.

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