ON SUMS OF FIBONACCI-TYPE RECIPROCALS

W. E. GREIG
West Virginia University, Morgantown, West Virginia 26506

Letting j be an integer, consider sequences of the form

(1) $$P_{n+1} = jP_n + P_{n-1},$$

where $P_0 = 0$. Without loss of generality take $P_1 = 1$. As an example think of 0, 1, 3, 10, 33, 109, $\cdots$. We can define the Lucas complement of (1) to be

(2) $$P_n^* = P_{n+1} + P_{n-1}.$$

The solution of these via the characteristic equation for the roots of [1] is well known. Let the roots of

(3) $$(q^2 - jq - 1) = 0$$

be a, b. The theory of equations tells us that $ab = -1$ and that $(a+b) = j$. This gives $(a-b) = 2a - j$. Using the initial conditions it can easily be shown following the method of Vorob'ev [1] that

(4) $$P_n = (a^n - b^n)/(a-b) \qquad \text{and} \qquad P_n^* = (a^n + b^n).$$

A few manipulations suffice to show that

(5) $$P_{j,2n} = P_{j,n}P_{j,n}^*$$

and using $(ab)^n = (-1)^n$ we can prove

(6) $$P_{j,2n-1} = P_{j,n-1}P_{j,n}^* - \cos(\pi n).$$

Although well known for the Fibonacci and Lucas sequences when $j = 1$, their validity when $j \neq 1$ has not been appreciated. Similarly we can derive

(7) $$P_{j,4n+1} = P_{j,2n+1}P_{2n}^* - 1.$$

Good [2] has derived the harmonic sum

$$\sum_{m=0}^{n} (1/F_b) = 3 - F_{B-1}/F_B,$$

where $b = 2^m$ and $B = 2^n$ have the virtue of conciseness. A double generalization follows introducing j as above and k a natural number arbitrary multiplier.

Theorem.

(8) $$\sum_{m=0}^{n} (1/P_{j,kb}) = C_{j,k} - P_{j,kB-1}/P_{j,kB} \quad \text{for} \quad n \geqslant 1.$$

Proof. Let j have any value, then as the basis for induction the proposition is certainly true for $n = 1$ since that merely defines the parameter $C_{j,k}$. Now assume that it is true for some $B = 2^n$ and add the next term $(1/P_{j,k2B})$ to each side. Hence the added term will equal the new minus the old right-hand side.

$$1/P_{j,k2B} = (P_{j,kB-1}/P_{j,kB}) - (P_{j,k2B-1}/P_{j,k2B}).$$

Cross-multiplying we have

$$P_{j,kB} = P_{j,2kB}P_{j,kB-1} - P_{j,2kB-1}P_{j,kB}$$

which is easy to prove using a Binet type of formula (4) as only the cross-product terms are non-zero. But it would be more aesthetically appealing to keep the proof in the realm of integers. This is easily done by substitution of first (5) and then (6) into the above equation. This completes the inductive transition.

Recall that $C_{j,k}$ is found from (8) when $n = 1$. The numerators of $C_{j,k}$ are thus

$$P_{j,2k}C_{j,k} = (1 + P^*_{j,k} + P_{j,2k-1}). \tag{9}$$

Successive application of (1) shows that

$$\{P_{j,k}\} = 0,\ 1,\ j,\ (j^2+1),\ (j^3+2j),\ (j^4+3j^2+1),\ (j^5+4j^3+3j),\ \cdots. \tag{10}$$

And using the definition (2) for the Lucas complement one finds

$$\{P^*_{j,k}\} = 2,\ j,\ (j^2+2),\ (j^3+3j),\ (j^4+4j+2),\ (j^5+5j^3+5j),\ \cdots. \tag{11}$$

And using (9) the numerators of $C_{j,k}$ are

$$\{P_{j,2k}C_{j,k}\} = 4, (j+2),\ (2j^2+4),\ (j^4+j^3+3j^2+3j+2),\ (j^6+6j^4+10j^2+4),$$
$$(j^8+7j^6+j^5+15j^4+5j^3+10j^2+5j+2),\ \cdots. \tag{12}$$

Table of $C_{j,k}$ Values

(written in the form with denominator $P_{j,2k}$ as in Eq. (9))

j/k	1	2	3	4	5	6
1	3/1	6/3	10/8	21/21	46/55	108/144
2	4/2	12/12	44/70	204/408	1068/2378	
3	5/3	22/33	146/360	1309/3927	13364/42837	
4	6/4	36/72	382/1292	5796/23188	99574/416020	
5	7/5	54/135	843/3640	19629/98145	513402/2646275	
j		$2/j$		$1/j$		

There are some simplifications. When $k \equiv 0 \pmod 4$ then using (5) gives $C_{j,k} = P_{j,kh-1}/P_{j,kh}$, where $h = ½$ and for $k = 4, 8, \cdots C_{j,k} = (1/j), (1/j - 1/P_{j,4}), \cdots$. When $k \equiv 2 \pmod 4$ then using (7) one finds

$$C_{j,k} = P^*_{j,kh-1}/P^*_{j,kh},$$

where $h = ½$ and for $k = 2, 6, \cdots C_{j,k} = 2/j, (1/j - 1/P^*_{j,3}), \cdots$. A short table of $C_{j,k}$ values is given and the interested reader can extend it with some patience.

Returning to the point of this paper, if we sum both sides of (8) over all odd k then the left-hand side is intuitively obviously a sum over all the natural numbers. The right-hand side of (8) is merely a sum over all odd k and so the sum of the reciprocals of numbers satisfying (1) (which I call coprime sequences) has been reduced to half the number of terms. The special case of Fibonacci numbers, $j = 1$, was derived by the author in October 1975 and is [3]. Gould [4, Eq. 2] expresses the rearrangement array as a sum and goes on to generalize it into partition arrays, his equation (9). So from (8) I write

$$\sum_{n=1}^{\infty} (1/P_{j,n}) = \sum_{k=1}^{\infty} (C_{j,k} - 1/a) \quad \text{for } k \text{ odd} \tag{13}$$

$$= \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} [1/P_{j,k} + 1/P_{j,2k} + (a-b)/(a^{4k}-1)] \tag{14}$$

as but two of several expressions that can be derived using Binet's expressions (4), where $a + b = j$ and $ab = -1$. All of the equations in the author's earlier paper [3] are valid here by merely replacing $\sqrt{5}$ by the more general $(a - b)$ and I do not see any point in taking up space to repeat them. I refer to sequences satisfying (1) as coprime sequences because they fulfill a generalization of a theorem in Vorob'ev [1] showing that only in this case are adjacent terms always coprime. The author used the generalization of this theorem in an earlier work [5]

REFERENCES

1. N. N. Vorob'ev, *Fibonacci Numbers,* Blaisdell, New York, 1961, pp. 18–20 and p. 30.
2. I. J. Good, "A Reciprocal Series of Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 12, No. 4 (Dec., 1974), p. 346.
3. W. E. Greig, "Sums of Fibonacci Reciprocals," *The Fibonacci Quarterly,* Vol. 15, No. 1 (Feb., 1977), pp. 46–48.
4. H. W. Gould, "A Rearrangement of Series Based on a Partition of the Natural Numbers," *The Fibonacci Quarterly,* Vol. 15, No. 1 (Feb., 1977), pp. 67–72.
5. W. E. Greig, "Bodė's Rule and Folded Sequences," *The Fibonacci Quarterly,* Vol. 14, No. 2 (April, 1976), p. 129.

****