GENERATING IDENTITIES FOR GENERALIZED FIBONACCI AND LUCAS TRIPLES

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BACKGROUND

In his article on generating identities for Pell triples, which involve the two Pell sequences, Serkland [5] modelled his arguments on those used by Hansen [1] for Fibonacci and Lucas sequences. Both articles suggest generalizations in a natural way.

Consider the following pairs of sequences (1) and (2), and (3) and (4):

				$n=0$	1	2	3	4	5	6	
(1)	Fibonacci	F_n	⋯	0	1	1	2	3	5	8	⋯
(2)	Lucas	L_n	⋯	2	1	3	4	7	11	18	⋯
(3)	Pell	P_n	⋯	0	1	2	5	12	29	70	⋯
(4)	Pell	R_n	⋯	2	2	6	14	34	82	198	⋯

for which the recurrence relations

(5) $$F_{n+2} = F_{n+1} + F_n$$

(6) $$L_{n+2} = L_{n+1} + L_n$$

(7) $$P_{n+2} = 2P_{n+1} + P_n$$

(8) $$R_{n+2} = 2R_{n+1} + R_n$$

and the summation relations

(9) $$F_{n+1} + F_{n-1} = L_n ,$$

(10) $$P_{n+1} + P_{n-1} = R_n$$

hold.

It is natural to examine pairs of sequences $\{A_n\}$ and $\{B_n\}$ similar to (1) and (2), and (3) and (4) having the properties:

(11) $$\begin{cases} \text{(i)} & A_0 = 0,\ A_1 = 1,\ A_{n+2} = cA_{n+1} + dA_n \quad (c \neq 0,\ d \neq 0) \\ \text{(ii)} & B_0 = 2,\ B_1 = c,\ B_{n+2} = cB_{n+1} + dB_n \\ \text{(iii)} & A_{n+1} + A_{n-1} = B_n \end{cases}$$

Thus, $A_n \equiv F_n$ and $B_n \equiv L_n$ if $c = 1, d = 1$, while $A_n \equiv P_n$ and $B_n \equiv R_n$ if $c = 2,\ d = 1$.

Generally, n is any integer. From (11) (i) and (ii), we may deduce that when $d = 1$,

(12) $$A_{-n} = (-1)^{n+1}A_n$$

(13) $$B_{-n} = (-1)^n B_n$$

(14) $$A_{-n+1} + A_{-n-1} = B_{-n} .$$

Result (14) may be readily derived from (11) (iii), (12) and (13).

It looks as though $d = 1$ is a condition for property (11) (iii), which generalizes (9) and (10), to exist. We proceed to establish this fact.

GENERALIZATIONS

The Binet forms for A_n and B_n are

$$A_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{15}$$

$$B_n = \alpha^n + \beta^n\ , \tag{16}$$

where α, β are the (distinct) roots of $x^2 - cx - d = 0$, so that

$$\alpha = \frac{c+D}{2},\quad \beta = \frac{c-D}{2},\quad \alpha+\beta = c,\quad \alpha - \beta = D,\quad D = \sqrt{c^2+4d},\quad \alpha\beta = -d. \tag{17}$$

From (11) (iii), (15) and (16), we have

$$(\alpha^{n+1} - \beta^{n+1}) + (\alpha^{n-1} - \beta^{n-1}) = (\alpha - \beta)(\alpha^n + \beta^n)$$

$$(\alpha^{n-1} - \beta^{n-1})(\alpha\beta + 1) = 0 \quad \text{on simplification}$$

$$\alpha\beta + 1 = 0 \quad \because \quad \alpha^{n-1} - \beta^{n-1} \neq 0 \quad (\text{i.e., } \alpha \neq \beta)$$

$$d = 1 \quad \because \quad \alpha\beta = -d \quad \text{by (17)}. \tag{18}$$

Thus, the required condition is $d = 1$ with c unrestricted.

Consequently, there are infinitely many pairs of sequences $\{A_n\}$ and $\{B_n\}$ having the properties:

$$\left\{\begin{array}{ll} \text{(i)} & A_0 = 0,\quad A_1 = 1,\quad A_{n+2} = cA_{n+1} + A_n \quad (c \neq 0) \\ \text{(ii)} & B_0 = 2,\quad B_1 = c,\quad B_{n+2} = cB_{n+1} + B_n \\ \text{(iii)} & A_{n+1} + A_{n-1} = B_n\ . \end{array}\right. \tag{11$'$}$$

Their Binet forms (15) and (16) now involve

$$\alpha = \frac{c+D}{2},\quad \beta = \frac{c-D}{2},\quad \alpha+\beta = c,\quad \alpha - \beta = D,\quad D = \sqrt{c^2+4},\quad \alpha\beta = -1, \tag{17$'$}$$

where α, β are now the roots of $x^2 - cx - 1 = 0$.

Some terms of these sequences are:

		⋯	$n = -3$	-2	-1	0	1	2	3	4	⋯
(19)	A_n	⋯	c^2+1	$-c$	1	0	1	c	c^2+1	c^3+2c	⋯
(20)	B_n	⋯	$-(c^3+3c)$	c^2+2	$-c$	2	c	c^2+2	c^3+3c	c^4+4c+2	⋯

Generating functions for these sequences are

$$\sum_{n=1}^{\infty} A_n x^n = x(1 - cx - x^2)^{-1} \tag{21}$$

$$\sum_{n=0}^{\infty} B_n x^n = (2 - cx)(1 - cx - x^2)^{-1} \tag{22}$$

The Theorems given in Serkland [5] follow directly for $\{A_n\}$ and $\{B_n\}$ by employing his methods, though in Theorems 1, 2, 3 use of the Binet forms (15) and (16) with (17)′ produces the results without difficulty.

Following Serkland's numbering [5], we have these generalized theorems:

Theorem 1. $$A_nB_m + A_{n-1}B_{m-1} = B_{m+n-1}$$

Theorem 2. $$A_nA_m + A_{n-1}A_{m-1} = A_{m+n-1}$$

Theorem 3. $$B_mB_n + B_{m-1}B_{n-1} = B_{m+n} + B_{m+n-2} = (c^2+4)A_{m+n-1}$$

Theorem 4. $$A_pA_qB_r = \sum_{k=0}^{q-1} (A_{k+1}B_{p+k+r-k} - A_{p+k+1}B_{q+r-k})$$

Theorem 5. $$A_pA_qA_r = \sum_{k=0}^{r-1} (A_{p+q+r-k}A_{k+1} - A_{p+k+1}A_{q+r-k})$$

Theorem 6. $$A_pB_qB_r = \sum_{k+0}^{p-1} ((c^2+4)A_{q+r+k+1}A_{p-k} - B_{q+k+1}B_{p+r-k})$$

Theorem 7. $$B_pB_qB_r = (c^2+4)\left[\sum_{k=0}^{p-2} (A_{q+r+k+1}B_{p-k} - A_{p+r-k}B_{q+k+1}) + cA_{p+q+r}\right] - cB_{p+q}B_{r+1}\,.$$

Of these theorems, we prove only the second part of Theorem 3 and all of Theorem 7 (taking the opportunity to correct some typographical errors in the original). A neater form for the expression of Theorem 3 (second part) is

$$B_{n+1} + B_{n-1} = (c^2+4)A_n$$

which should be compared with 11 (iii).

Proof of Theorem 3 (second part).

$$\begin{aligned}
B_{m+n} + B_{m+n-2} &= (A_{m+n+1} + A_{m+n-1}) + (A_{m+n-1} + A_{m+n-3}) && \text{by (11)' (iii)}\\
&= A_{m+n+1} + 2A_{m+n-1} + A_{m+n-3}\\
&= (cA_{m+n} + A_{m+n-1}) + 2A_{m+n-1} + A_{m+n-3} && \text{by (11)' (i)}\\
&= cA_{m+n} + 3A_{m+n-1} + A_{m+n-3}\\
&= c(cA_{m+n-1} + A_{m+n-2}) + 3A_{m+n-1} + A_{m+n-3} && \text{by (11)' (i)}\\
&= (c^2+3)A_{m+n-1} + (cA_{m+n-2} + A_{m+n-3})\\
&= (c^2+4)A_{m+n-1} && \text{by (11)' (i)}\,.
\end{aligned}$$

Proof of Theorem 7.

$$\begin{aligned}
B_pB_qB_r &= (A_{p+1} + A_{p-1})B_qB_r && \text{by (11)' (iii)}\\
&= A_{p+1}B_qB_r + A_{p-1}B_qB_r = \sum_{k=0}^{p} ((c^2+4)A_{q+r+k+1}A_{p-k+1} - B_{q+k+1}B_{p+r-k+1})\\
&\quad + \sum_{k=0}^{p-2} ((c^2+4)A_{q+r+k+1}A_{p-k-1} - B_{q+k+1}B_{p+r-k-1}) && \text{by Theorem 6}\\
&= \sum_{k=0}^{p-2} \left[(c^2+4)A_{q+r+k+1}(A_{p-k+1} + A_{p-k-1}) - B_{q+k+1}(B_{p+r-k+1} + B_{p+r-k-1})\right]\\
&\quad + (c^2+4)A_2A_{p+q+r} - B_{p+q}B_{r+2} + (c^2+4)A_1A_{p+q+r+1} - B_{p+q+1}B_{r+1}\\
&= \sum_{k=0}^{p-2} (c^2+4)(A_{q+r+k+1}B_{p-k} - B_{q+k+1}A_{p+r-k})\\
&\quad + (c^2+4)(cA_{p+q+r} + A_{p+q+r+1}) - (B_{p+q}B_{r+2} + B_{p+q+1}B_{r+1}) && \text{by (11)' (iii), (19), and Theorem 3}\\
&= (c^2+4)\left[\sum_{k=0}^{p-2} (A_{q+r+k+1}B_{p-k} - B_{q+k+1}A_{p+r-k}) + cA_{p+q+r} + A_{p+q+r+1}\right]\\
&\quad - (cB_{p+q}B_{r+1} + B_{p+q}B_r + B_{p+q+1}B_{r+1}) =
\end{aligned}$$

$$= (c^2+4)\left[\sum_{k=0}^{p-2}(A_{q+r+k+1}B_{p-k} - A_{p+r-k}B_{q+k+1}) + cA_{p+q+r}\right] - cB_{p+q}B_{r+1} \quad \text{by Theorem 3.}$$

Putting $c = 1$ in Theorems 1–7 we obtain the theorems of Hansen [1] for the Fibonacci-Lucas pair of sequences. With $c = 2$, the theorems of Serkland [5] for the two Pell sequences follow. The forms of Hansen's Theorem 5 and Serkland's Theorem 5 should be compared.

The natural extension of the special cases considered by Hansen [1] and Serkland [5] occurs when $c = 3$. Call these sequences $\{X_n\}$ and $\{Y_n\}$, some terms of which are:

		...	$n=-3$	-2	-1	0	1	2	3	4	5	6	...
(23)	X_n	...	10	-3	1	0	1	3	10	33	109	360	...
(24)	Y_n	...	-36	11	-3	2	3	11	36	119	393	1298	...

Theorems 1–7, and the associated background details, readily apply with $c = 3$ $(c^2 + 4 = 13)$. Interested readers may construct other pairs of related sequences from the infinitely many possibilities manifested in (19) and (20).

CONCLUDING REMARKS

Examples of familiar pairs of sequences which are excluded from our considerations (i.e., for which $d \neq 1$) are

(a) the Fermat sequences $\{2^n - 1\}$, $\{2^n + 1\}$ $(c = 3,\ d = -2)$

(b) the Chebyshev sequences

$$\{T_n = 2\cos n\theta\},\quad \left\{U_n = \frac{\sin (n+1)\theta}{\sin\theta}\right\}\quad (c = 2\cos\theta,\ d = -1).$$

(Obviously, in (a), $2^n + 1 = (2^n - 1) + 2$, i.e., the two Fermat sequences are not independent of each other.)

Comments on the excluded degenerate case which occurs when $\alpha = \beta$, i.e., $D = \sqrt{c^2 + 4d} = 0$, may be found in Horadam [3].

Further information on the Pell sequences, as special cases of the sequence $\{W_n\}$ for which

$$W_0 = a,\quad W_1 = b,\quad W_{n+2} = cW_{n+1} + dW_n$$

(which generalizes (11) (i) and (ii)), is given in Horadam [4]. For a partition of $\{W_n\}$ into Fibonacci-type and Lucas-type sequences the reader is referred to Hilton [2], which is generalized to r^{th}-order sequences by Shannon [6].

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