FORMULA DEVELOPMENT THROUGH FINITE DIFFERENCES

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FINITE DIFFERENCE CONCEPT

Given a function $f(n)$ the first difference of the function is defined

$$\Delta f(n) = f(n+1) - f(n).$$

(NOTE: There is a more generalized finite difference involving a step of size h but this can be reduced to the above by a linear transformation.)

EXAMPLES

$$f(n) = 5n+3, \qquad \Delta f(n) = 5(n+1)+3-(5n+3) = 5$$

$$f(n) = 3n^2+7n+2 \mid \Delta f(n) = 3(n+1)^2+7(n+1)+2-(3n^2+7n+2) = 6n+10.$$

Finding the first difference of a polynomial function of higher degree involves a considerable amount of arithmetic. This can be reduced by introducing a special type of function known as a generalized factorial.

GENERALIZED FACTORIAL

A generalized factorial

$$(x)^{(n)} = x(x-1)(x-2)\cdots(x-n+1),$$

where there are n factors each one less than the preceding. To tie this in with the ordinary factorial note that

$$n^{(n)} = n!$$

EXAMPLE

$$x^{(4)} = x(x-1)(x-2)(x-3).$$

The first difference of $x^{(n)}$ is found as follows:

$$\begin{aligned}\Delta x^{(n)} &= (x+1)x(x-1)\cdots(x-n+3)(x-n+2) - x(x-1)(x-2)\cdots(x-n+2)(x-n+1)\\ &= x(x-1)(x-2)\cdots(x-n+3)(x-n+2)[x+1-(x-n+1)] = nx^{(n-1)}.\end{aligned}$$

Note the nice parallel with taking the derivative of x^n in calculus.

To use the factorial effectively, in working with polynomials we introduce Stirling numbers of the first and second kind. Stirling numbers of the first kind are the coefficients when we express factorials in terms of powers of x. Thus

$$x^{(1)} = x, \quad x^{(2)} = x(x-1) = x^2-x, \quad x^{(3)} = x(x-1)(x-2)(x-3) = x^3-3x^2+2x$$

$$x^{(4)} = x(x-1)(x-2)(x-3) = x^4-6x^3+11x^6-6x.$$

Stirling numbers of the first kind merely record these coefficients in a table.

Stirling numbers of the second kind are coefficients when we express the powers of x in terms of factorials.

$$x = x^{(1)}$$

$$x^2 = x^2-x+x = x^{(2)}+x^{(1)}$$

$$x^3 = x^3-3x^2+2x+(3x^2-3x)+x = x^{(3)}+3x^{(2)}+x^{(1)}$$

As one example of the use of these numbers let us find the difference of the polynomial function

$$4x^5-7x^4+9x^3-5x^2+3x-1.$$

Using the Stirling numbers of the second kind we first translate into factorials,

TABLE OF STIRLING NUMBERS OF THE FIRST KIND

power of x

n	1	2	3	4	5	6	7	8	9	10
1	1									
2	−1	1								
3	2	−3	1							
4	−6	11	−6	1						
5	24	−50	35	−10	1					
6	−120	274	−225	85	−15	1				
7	720	−1764	1624	−735	175	−21	1			
8	−5040	13068	−13132	6769	−1960	322	−28	1		
9	40320	−109584	118124	−67284	22449	−4536	546	−36	1	
10	−362880	1026576	−1172700	723680	−269325	63273	−9450	870	−45	1

TABLE OF STIRLING NUMBERS OF THE SECOND KIND

Coefficients of $x^{(k)}$

n \ k	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	3	1							
4	1	7	6	1						
5	1	15	25	10	1					
6	1	31	90	65	15	1				
7	1	63	301	350	140	21	1			
8	1	127	966	1701	1050	266	28	1		
9	1	255	3025	7770	6951	2646	462	36	1	
10	1	511	9330	34105	42525	22827	5880	750	45	1

TABLE OF FACTORIALS

	$x^{(5)}$	$x^{(4)}$	$x^{(3)}$	$x^{(2)}$	$x^{(1)}$	c
$4x^5$	4	40	100	60	4	
$-7x^4$		−7	−42	−49	−7	
$9x^3$			9	27	9	
$-5x^2$				−5	−5	
$3x-1$					3	−1

Giving

$$4x^{(5)} + 33x^{(4)} + 67x^{(3)} + 33x^{(2)} + 4x^{(1)} - 1.$$

Using the formula for finding the difference of a factorial the first difference is given by

$$20x^{(4)} + 132x^{(3)} + 201x^{(2)} + 66x^{(1)} + 4.$$

Now we translate back to a polynomial function by using Stirling numbers of the first kind.

	x^4	x^3	x^2	x	c
$20x^{(4)}$	20	−120	220	−120	
$132x^{(3)}$		132	−396	264	
$201x^{(2)}$			201	−201	
$66x^{(1)} + 4$				66	4

The resulting polynomial function is

$$20x^4 + 12x^3 + 25x^2 + 9x + 4$$

A POLYNOMIAL FUNCTION FROM TABULAR VALUES

From the above it is evident that the first difference of a polynomial of degree n is a polynomial of degree $n-1$; the second difference is a polynomial of degree $n-2$; etc., so that the n^{th} difference is a constant. The $(n+1)^{st}$ difference is zero. As a matter of fact since at each step we multiply the coefficient of the first term by the power of x, the n^{th} difference of

$$a_0x^n + a_1x^{n-1} + a_2x^{n-3} + \cdots + a_{n-1}x + a_n$$

is $a_0n!$

Conversely if we have a table of values and find that the r^{th} difference is a constant we may conclude that these values fit a polynomial function of degree r. For example for

$$f(x) = 5x^3 - 7x^2 + 3x - 8$$

we have a table of values and finite differences as follows.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	−8			
		1		
1	−7		16	
		17		30
2	10		46	
		63		30
3	73		76	
		139		30
4	212		106	
		245		30
5	457		136	
		381		30
6	838		166	
		547		
7	1385			

The problem is how to arrive at the original formula from this table.

Suppose that the polynomial is expressed in terms of factorials with undetermined coefficients $b_0, b_1, b_2, \cdots$. The problem will be solved if we find these coefficients.

$$f(x) = b_0 + b_1x^{(1)} + b_2x^{(2)} + b_3x^{(3)} + b_4x^{(4)} + b_5x^{(5)} + \cdots$$
$$\Delta f(x) = b_1 + 2b_2x^{(1)} + 3b_3x^{(2)} + 4b_4x^{(3)} + 5b_5x^{(4)} + \cdots$$
$$\Delta^2 f(x) = 2!b_2 + 3*2b_3x^{(1)} + 4*3b_4x^{(2)} + 5*4b_5x^{(3)} + \cdots$$
$$\Delta^3 f(x) = 3!b_3 + 4*3*2b_4x^{(1)} + 5*4*3b_5x^{(2)} + \cdots$$
$$\Delta^4 f(x) = 4!b_4 + 5*4*3*2b_5x^{(1)} + \cdots .$$

Set $x = 0$. Since any factorial is zero for $x = 0$ we have from the above:

$$f(0) = b_0 \quad \text{or} \quad b_0 = f(0)$$
$$\Delta f(0) = b_1 \quad \text{or} \quad b_1 = \Delta f(0)$$
$$\Delta^2 f(0) = 2!b_2 \quad \text{or} \quad b_2 = \Delta^2 f(0)/2!$$
$$\Delta^3 f(0) = 3!b_3 \quad \text{or} \quad b_3 = \Delta^3 f(0)/3!$$
$$\Delta^4 f(0) = 4!b_4 \quad \text{or} \quad b_4 = \Delta^4 f(0)/4!.$$

Hence

$$f(x) = f(0) + \Delta f(0)x^{(1)} + \Delta^2 \frac{f(0)}{2!}x^{(2)} + \Delta^3 \frac{f(0)}{3!}x^{(3)} + \Delta^4 \frac{f(0)}{4!}x^{(4)} +$$

This is known as Newton's forward difference formula. We can find the quantities $f(0)$, $\Delta f(0)$, $\Delta^2 f(0)$, $\Delta^3 f(0)$, $\Delta^4 f(0)$, $\cdots$ from the top edge of our numerical table of values provided the first value in our table is 0.

$$f(x) = -8 + x + 16x^{(2)}/2! + 30x^{(3)}/3! = -8 + x + 8x^2 - 8x + 5x^3 - 15x^2 + 10x = 5x^3 - 7x^2 + 3x - 8.$$

Stirling numbers of the first kind can be used in this evaluation.

SUMMATIONS INVOLVING POLYNOMIAL FUNCTIONS

Since a polynomial function can be expressed in terms of factorials it is sufficient to find a formula for summing any factorial. More simply by dividing the k^{th} factorial by $k!$ we have a binomial coefficient and the summation of these coefficients leads to a beautifully simple sequence of relations.

To evaluate

$$\sum_{k=1}^{n} k, \quad \text{let} \quad \sum_{k=1}^{n} k = \varphi(n)$$

meaning that the value is a function of n. Then

$$\Delta\varphi(n) = \sum_{k=1}^{n+1} k - \sum_{k=1}^{n} k = n+1.$$

Now $\Delta n = 1$ and $\Delta n^{(2)}/2 = n$. Hence

$$\varphi(n) = \sum_{k=1}^{n} k = n^{(2)}/2 + n + C = n(n+1)/2 + C,$$

where the C is necessary in taking the anti-difference since the difference of a constant is zero. This corresponds to the constant of integration in the indefinite integral. To find the value of C let $n = 1$. Then

$$1 = 1*2/2 + C \quad \text{so that} \quad C = 0.$$

Hence

$$\sum_{k=1}^{n} k = n(n+1)/2 = \binom{n+1}{2}$$

a well-known formula. Next, let

$$\sum_{k=1}^{n} \binom{k+1}{2} = \varphi(n), \quad \Delta\varphi(n) = \sum_{k=1}^{n+1} \binom{k+1}{2} - \sum_{k=1}^{n} \binom{k+1}{2} = \binom{n+2}{2}$$

The difference

$$\Delta\binom{n+2}{3} = \binom{n+2}{2}.$$

Hence

$$\varphi(n) = \sum_{k=1}^{n} \binom{k+1}{2} = \binom{n+2}{3} + C.$$

$n = 1$ shows that $C = 0$. The sequence of formulas can be continued:

$$\sum_{k=1}^{n} \binom{k+2}{3} = \binom{n+3}{4}$$

and in general

$$\sum_{k=1}^{n} \binom{k+r}{r+1} = \binom{n+r+1}{r+2}.$$

One could derive the formula for the summation of a factorial from the above but proceeding directly:

$$\sum_{k=1}^{n} k^{(r)} = \varphi(n), \qquad \Delta\varphi(n) = (n+1)^{(r)}.$$

Hence,

$$\varphi(n) = \sum_{k=1}^{n} k^{(r)} = \frac{(n+1)^{(r+1)}}{r+1} + C.$$

Taking $n = r$,

$$r! = (r+1)^{(r+1)}/(r+1) + C$$

so that $C = 0$.

$$\sum_{k=1}^{n} k^{(r)} = \frac{(n+1)^{(r+1)}}{r+1} .$$

Again there is a noteworthy parallel with the integral calculus in this formula.

For examples we take some formulas from L. B. W. Jolley, *Summation of Series,*

EXAMPLE 1. (45) p. 8, $\displaystyle\sum_{k=1}^{n} (3k-1)(3k+2) = 2*5 + 5*8 + 8*11 + \cdots.$

This equals

$$\sum_{k=1}^{n} (9k^2 + 3k - 2) = \sum_{k=1}^{n} (9k^{(2)} + 12k^{(1)} - 2) = 9\,\frac{(n+1)^{(3)}}{3} + 12\,\frac{(n+1)^{(2)}}{2} - 2(n+1) + C.$$

Taking $n = 1$, $2*5 = 6*2 - 2*2 + c$ so that $C = 2$

$$\sum_{k=1}^{n} (3k-1)(3k+2) = 3n^3 - 3n + 6n^2 + 6n - 2n - 2 + 2 = n(3n^2 + 6n + 1).$$

EXAMPLE 2. (50) p. 10

$$\sum_{k=1}^{n} k(k+3)(k+6) = 1*4*7 + 42*5*k + 3*6*9 + \cdots = \sum_{k=1}^{n} (k^3 + 9k^2 + 18k) = \sum_{k=1}^{n} (k^{(3)} + 12k^{(2)} + 28k^{(1)})$$

$$= \frac{(n+1)^{(4)}}{4} + 12\,\frac{(n+1)^{(3)}}{3} + 28\,\frac{(n+1)^{(2)}}{2} + C = \frac{(n+1)n}{4}\,[(n-1)(n-2) + 16(n-1) + 56] + C$$

$$= n(n+1)(n+6)(n+7)/4 + C.$$

Setting $n = 1$, $1*4*7 = 1*2*7*8/4 + c$ so that $C = 0$

$$\sum_{k=1}^{n} k(k+3)(k+6) = n(n+1)(n+6)(n+7)/4.$$

EXAMPLE 3. (49) p. 10

$$\sum_{k=1}^{n} (3k-2)(3k+1)(3k+4) = 1*4*7 + 4*7*10*13 + \cdots .$$

This can be changed directly into a factorial:

$$27\sum_{k=1}^{n} (k - 2/3)(k + 1/3)(k + 4/3) = 27\sum_{k=1}^{n} (k + 4/3)^{(3)}$$

giving

$$27(n + 7/3)^{(4)}/4 + C = (3n+7)(3n+4)(3n+1)(3n-2)/12 + C.$$

Setting $n = 1$, $28 = (10*7*4*1)/12 + C$ so that $C = 56/12$

$$(3k-2)(3k+1)(3k+4) = (3n+7)(3n+4)(3n+1)(3n-2)/12 + 56/12\,.$$

SUMMATIONS THROUGH NEGATIVE FACTORIALS

Starting with the relation

$$x^{(m)} * (x-m)^{(n)} = x^{(m+n)}$$

set $m = -n$.

$$x^{(-n)} * (x+n)^{(n)} = x^{(0)} = 1.$$

Therefore $x^{(-n)} = 1/(x+n)^{(n)}$.

Possibly this bit of mathematical formalism seems unconvincing. Suppose then we define the negative factorial in this fashion.

$$\begin{aligned}\Delta x^{(-n)} &= 1/[(x+n+1)(x+n)(x+n-1)\cdots(x+2)] - 1/[(x+n)(x+n-1)(x+n-2)\cdots(x+2)(x+1)]\\ &= 1/[(x+n)(x+n-1)\cdots(x+2)][1/(x+n+1) - 1/(x+1)]\\ &= -n/[(x+n+1)(x+n)(x+n-1)\cdots(x+1)] = -nx^{(-n-1)}\end{aligned}$$

showing that the difference relation that applies to positive factorials holds as well for negative factorials defined in this fashion. Consequently the anti-difference which is used in finding the value of summations can be employed with negative factorials apart from the case of -1.

EXAMPLE 1.

$$\sum_{k=1}^{n} 1/[k(k+1)(k+2)] = \sum_{k=1}^{n} (k-1)^{(-3)} = n^{(-2)}/(-2) + C = -1/[2(n+2)(n+1)] + C\,.$$

Setting $n = 1$, $1/6 = -1/(2*3*2) + C$, so that $C = 1/4$

$$\sum_{k=1}^{n} 1/[k(k+1)(k+2)] = 1/4 - 1/[2(n+2)(n+1)]\,.$$

EXAMPLE 2. Jolley, No. 210, p. 40

$$\begin{aligned}\sum_{k=1}^{n} 1/[(3k-2)(3k+1)(3k+4)] &= (1/27)\sum_{k=1}^{n} 1/[k-2/3)(k+1/3)(k+4/3)] = (1/27)\sum_{k=1}^{n} (k-5/3)^{(-3)}\\ &= (1/27)(n-2/3)^{(-2)}/(-2) + C = -1/[6(3n+4)(3n+1)] + C\,.\end{aligned}$$

Setting $n = 1$, $1/(1*4*7) = -1/(6*7*4) + C;\ C = 1/24$

$$\sum_{k=1}^{n} 1/[(3k-2)(3k+1)(3k+4)] = 1/24 - 1/[6(3n+4)(3n+1)]$$

EXAMPLE 3. Jolley, No. 213, p. 40

$$\sum_{k=1}^{n} (2k-1)/[k(k+1)(k+2)] = 2\sum_{k=1}^{n} 1/[(k+1)(k+2)] - \sum_{k=1}^{n} 1/[k(k+1)(k+2)]\,.$$

The second summation was evaluated in Example 1. The first gives

$$2\sum_{k=1}^{n} k^{(-2)} = 2(n+1)^{(-1)}/(-1) + C\,.$$

Altogether, the result is

$$-2/(n+2) - 1/4 + 1/[2(n+2)(n+1)] + C\,.$$

Setting $n = 1$, $1/6 = -2/3 - 1/4 + 1/12 + C$ so that $C = 1$

$$\sum_{k=1}^{n} (2k-1)/[k(k+1)(k+2)] = 3/4 - 2/(n+2) + 1/[2(n+2)(n+1)]\,.$$

DIFFERENCE RELATION FOR A PRODUCT

Let there be two functions $f(n)$ and $g(n)$. Then

$$\Delta f(n)g(n) = f(n+1)g(n+1) - f(n)g(n) = f(n+1)g(n+1) - f(n+1)g(n) + f(n+1)g(n) - f(n)g(n)$$
$$= f(n+1)\Delta g(n) + g(n)\Delta f(n)\,.$$

This will be found useful in a variety of instances.

SUMMATIONS INVOLVING GEOMETRIC PROGRESSIONS

A geometric progression with terms ar^{k-1} can be summed as follows:

$$\sum_{k=1}^{n} ar^{k-1} = \varphi(n), \qquad \Delta\varphi(n) = ar^n$$

But $\Delta r^n = r^{n+1} - r^n = r^n(r-1)$. Hence

$$\varphi(n) = \sum_{k=1}^{n} ar^{k-1} = \Delta^{-1}(ar^n) = ar^n/(r-1) + C\,.$$

Setting $n = 1$, $a = ar/(r-1) + C$ so that $C = -a/(r-1)$. Hence,

$$\sum_{k=1}^{n} ar^{k-1} = a(r^n - 1)/(r-1)\,.$$

The summation

$$\sum_{k=1}^{n} kr^k = \varphi(n), \qquad \Delta\varphi(n) = (n+1)r^{n+1}, \qquad \Delta(nr^{n+1}) = (n+1)r^{n+1}(r-1) + r^{n+1}$$

using the product formula on page 8 with the first function as n and the second as r^{n+1}.

$$(n+1)r^{n+1} = \Delta\,[nr^{n+1}/(r-1)] - r^{n+1}/(r-1)\,.$$

Hence

$$\Delta^{-1}(n+1)r^{n+1} = nr^{n+1}/(r-1) - r^{n+1}/(r-1)^2 + C.$$

Setting $n = 1$, $r = r^2/(r-1) - r^2/(r-1)^2 + C$; $C = r/(r-1)^2$. Accordingly

$$\sum_{k=1}^{n} kr^k = nr^{n+1}/(r-1) - r^{n+1}/(r-1)^2 + r/(r-1)^2\,.$$

EXAMPLE.
$$\sum_{k=1}^{5} k*3^k = 1*3 + 2*9 + 3*27 + 4*81 + 5*243 = 1641\,.$$

By formula
$$5*3^6/2 - 3^6/4 + 3/4 = 1641\,.$$

FIBONACCI SUMMATIONS

A Fibonacci sequence is defined by two initial terms T_1 and T_2 accompanied by the recursion relation

$$T_{n+1} = T_n + T_{n-1}\,.$$

SUM OF THE TERMS OF THE SEQUENCE

$$\sum_{k=1}^{n} T_k = \varphi(n), \qquad \Delta\varphi(n) = T_{n+1}, \qquad \Delta T_n = T_{n+1} - T_n = T_{n-1}\,.$$

Accordingly

$$\sum_{k=1}^{n} T_k = T_{n+2} + C\,.$$

Setting $n = 1$, $T_1 = T_3 + C$ or $C = T_1 - T_3 = -T_2$

$$\sum_{k=1}^{n} T_k = T_{n+2} - T_2$$

SUM OF THE SQUARES OF THE TERMS

$$\sum_{k=\alpha}^{n} T_k^2 = \varphi(n), \quad \Delta\varphi(n) = T_{n+1}^2 .$$

The anti-difference bears a strong resemblance to integration in the differential calculus. Just as we know integrals on the basis of differnetiation so likewise we find anti-differences on the basis of differences. Thus we try various expressions to see whether we can find one whose difference is the square of T_{n+1}.

$$\Delta T_n T_{n+1} = T_{n+1}T_{n+2} - T_n T_{n+1} = T_{n+1}(T_{n+2} - T_n) = T_{n+1}^2 .$$

Hence

$$\sum_{k=\alpha}^{n} T_k^2 = T_n T_{n+1} + C .$$

Setting $n = a$, $T_a^2 = T_a T_{a+1} + C$

$$C = T_a(T_a - T_{a+1}) = -T_a T_{a-1}, \quad \sum_{k=\alpha}^{n} T_k^2 = T_n T_{n+1} - T_a T_{a-1} .$$

SUMMATION OF ALTERNATE TERMS

$$\sum_{k=m}^{n} T_{2k+a} = \varphi(n), \quad \Delta\varphi(n) = T_{2(n+1)+a}, \quad \Delta T_{2n+a} = T_{2n+2+a} - T_{2n+a} = T_{2n+1+a} .$$

Hence

$$\Delta^{-1} T_{2(n+1)+a} = T_{2n+1+a} + C, \quad \sum_{k=m}^{n} T_{2k+a} = T_{2n+1+a} + C .$$

Setting $k = m$,

$$T_{2m+a} = T_{2m+1+a} + C, \qquad \sum_{k=m}^{n} T_{2k+a} = T_{2n+1+a} - T_{2m-1+a}$$

SUM OF EVERY FOURTH TERM

$$\sum_{k=1}^{n} T_{4k+a} = \varphi(n), \qquad \Delta\varphi(n) = T_{4n+4+a}$$

$$\Delta T_{4n+a} = T_{4n+4+a} - T_{4n+a} = T_{4n+3+a} + T_{4n+2+a} - T_{4n+2+a} + T_{4n+1+a} = T_{4n+3+a} + T_{4n+1+a}$$

To meet this situation we introduce a quantity

$$V_n = T_{n-1} + T_{n+1} .$$

Now

$$V_{n-1} + V_{n+1} = T_{n-2} + T_n + T_n + T_{n+2} = -T_{n-1} + T_n + 2T_n + T_n + T_{n+1} = 5T_n .$$

To obtain a difference which gives T we start with V. By a process similar to that for T

$$\Delta V_{4n+a} = V_{4n+3+a} + V_{4n+1+a} = 5T_{4n+2+a} .$$

Consequently,

$$\Delta^{-1} T_{4n+4+a} = (V_{4n+2+a})/5 + C = \sum_{k=1}^{n} T_{4k+a} .$$

Setting $n = 1$,

$$C = T_{4+a} - V_{6+a}/5, \quad \sum_{k=1}^{n} T_{4k+a} = (T_{4n+1+a} + T_{4n+3+a})/5 - (T_{5+a} + T_{7+a})/5 + T_{4+a} .$$

EXAMPLE. We use the terms of the sequence beginning 1,4.

1, 4, 5, 9, 14, 23, 37, 60, 97, 157, 254, 411, 665, 1076, 1741, 2817, 4558, 7375, 11933, 19308, 31241, 50549, 81790, 132339, 214129, 346468, 560597, 907065, 1467662, 2374727.

Let $a = 2$.

$$\sum_{k=1}^{5} T_{4k+2} = T_6 + T_{10} + T_{14} + T_{18} + T_{22} = 23 + 157 + 1076 + 7375 + 50549 = 59180.$$

By formula we have

$$(T_{23} + T_{25})/5 - (T_7 + T_9)/5 + T_6 = (81790 + 214129)/5 - (37 + 97)/5 + 23 = 59180.$$

SEQUENCE WITH ALTERNATING SIGNS

$$\sum_{k=m}^{n} (-1)^k T_{2k+a} = \varphi(n), \quad \Delta\varphi(n) = (-1)^{n+1} T_{2n+2+a}, \quad V_{2n+a} = T_{2n+1+a} + T_{2n-1} + a$$

$$\Delta(-1)^n V_{2n+a} = (-1)^{n+1} V_{2n+2+a} - (-1)^n V_{2n+a} = (-1)^{n+1} [V_{2n+2+a} + V_{2n+a}] = (-1)^{n+1} 5T_{2n+1+a}.$$

Hence

$$\sum_{k=m}^{n} (-1)^n T_{2k+a} = (-1)^n (V_{2n+1+a})/5 + C = (-1)^n [T_{2n+a} + T_{2n+a+2}]/5 + C.$$

Let $n = m$.

$$(-1)^m T_{2m+a} = (-1)^m [T_{2m+a} + T_{2m+2+a}]/5 + C$$

$$\sum_{k=m}^{n} (-1)^k T_{2k+a} = (-1)^n [T_{2n+a} + T_{2n+a+2}]/5 + (-1)^{m+1} [T_{2m+a} + T_{2m+a+2}]/5 + (-1)^m T_{2m+a}.$$

Using the 1,4 sequence once more

$$\sum_{k=3}^{7} (-1)^k T_{2k+3} = -T_9 + T_{11} - T_{13} + T_{15} - T_{17} = -97 + 254 - 665 + 1741 - 4558 = -3325.$$

By formula we have

$$-(T_{17} + T_{19})/5 + (T_9 + T_{11})/5 - T_9 = -(4558 + 11933)/5 + (97 + 254)/5 - 97 = -3325.$$

GEOMETRIC-FIBONACCI SUMS

POWER of 2.

$$\sum_{k=1}^{n} 2^k T_k = \varphi(n); \qquad \Delta\varphi(n) = 2^{n+1} T_{n+1}$$

$$\Delta 2^n T_n = 2^{n+1} T_n + 2^n T_n = 2^n (2T_{n-1} + T_n) = 2^n V_n,$$

where we have used the product relation on page 8 and introduced the sequence defined by

$$V_n = T_{n-1} + T_{n+1}.$$

Since $\Delta 2^n V_n = 5 * 2^n T_n$ (following the same steps as for T_n)

$$\varphi(n) = \Delta^{-1}(2^{n+1} T_{n+1}) = 2^{n+1} V_{n+1}/5 + C.$$

Setting $n = 1$, $2T_1 = 4V_2/5 + C$. Hence

$$\sum_{k=1}^{n} 2^k T_k = 2^{n+1}(T_n + T_{n+2})/5 + (6T_1 - 4T_3)/5.$$

EXAMPLE.

$$\sum_{k=1}^{5} 2^k T_k = 2*1 + 4*4 + 8*5 + 16*9 + 32*14 = 650 \text{ (1,4 sequence).}$$

By formula $[2^6(14+37)+6-4*5]/5 = 650.$

THE SUMMATION

$$\sum_{k=1}^{n} r^k T_k .$$

The direct approach leads to an apparent impasse. We wish to find the inverse difference of $r^{n+1}T_{n+1}$. Assume that it is of the form

$$A[r^k T_{n+1} + r^j T_n] .$$

This approach parallels what is done in the solution of differential equations. k, j, and A are undetermined constants. Taking the difference and setting it equal to $r^{n+1}T_{n+1}$ we have

$$A[r^{k+1}T_n + r^{j+1}T_{n-1} + r^k(r-1)T_{n+1} + r^j(r-1)T_n] = r^{n+1}T_{n+1} .$$

Replacing T_{n-1} on the left-hand side by $T_{n+1} - T_n$ and equating coefficients of T_{n+1} and T_n gives:

$$A[r^k(r-1) + r^{j+1}] = r^{n+1}, \qquad r^{k+1} + r^j(r-1) - r^{j+1} = 0.$$

From the second $j = k+1$. Then the first gives

$$A[r^{k+1} - r^k + r^{k+2}] = r^{n+1} .$$

Letting $k = n+1$ and $A = 1/(r^2+r-1)$ establishes equality. Hence

$$\sum_{k=1}^{n} r^k T_k = (r^{n+1}T_{n+1} + r^{n+2}T_n)/(r^2+r-1) + C, \quad C = (-r^2T_0 - rT_1)/(r^2+r-1)$$

$$\sum_{k=1}^{n} r^k T_k = [r^{n+1}T_{n+1} + r^{n+2}T_n - r^2T_0 - rT_1]/(r^2+r-1) .$$

EXAMPLE (1,4 sequence)

$$\sum_{k=1}^{5} 3^k T_k = 3*1 + 3^2*4 + 3^3*5 + 3^4*9 + 3^5*14 = 4305 .$$

By formula,

$$(3^6*23 + 3^7*14 - 27 - 3)/11 = 4305 .$$

FIBONACCI-FACTORIAL SUMMATIONS

THE SUMMATION

$$\sum_{k=1}^{n} kT_k = \varphi(n)$$

$$\Delta\varphi(n) = (n+1)T_{n+1}$$

$$\Delta nT_n = (n+1)T_{n-1} + T_n$$

$$\Delta nT_{n+2} = (n+1)T_{n+1} + T_{n+2}$$

$$\Delta^{-1}(n+1)T_{n+1} = nT_{n+2} - T_{n+3} + T_3 + C = \sum_{k=1}^{n} kT_k$$

in which we have used the formula

$$\Delta^{-1}T_{n+2} = T_{n+3} - T_3$$

$n = 1$ gives

$$T_1 = T_3 - T_4 + T_3 + C; \quad C = 0$$

so that

$$\sum_{k=1}^{n} kT_k = nT_{n+2} - T_{n+3} + T_3 .$$

Note that this is also $\Delta^{-1}(n+1)T_{n+1}$, a fact that is used in the next derivation.

EXAMPLE (1,4 sequence)

$$\sum_{k=1}^{5} kT_k = 1*1 + 2*4 + 3*5 + 4*9 + 5*14 = 130 .$$

By formula $5*36 - 60 + 5 = 130$.

THE SUMMATION

$$\sum_{k=1}^{n} k^{(2)}T_k = \varphi(n)$$

$$\Delta\varphi(n) = (n+1)^{(2)}T_{n+1}$$

$$\Delta n^{(2)}T_{n+2} = (n+1)^{(2)}T_{n+1} + 2nT_{n+2}$$

$$\sum_{k=1}^{n} k^{(2)}T_k = n^{(2)}T_{n+2} - 2(n-1)T_{n+3} + 2T_{n+4} - 2T_4 + C$$

in which the formula for the previous case was used.

For $n = 2$,

$$2T_2 = 2T_4 - 2T_5 + 2T_6 - 2T_4 + C; \qquad C = -2T_3$$

$$\sum_{k=1}^{n} k^{(2)}T_k = n^{(2)}T_{n+2} - 2(n-1)T_{n+3} + 2T_{n+4} - 2T_4 - 2T_3$$

VERIFICATION (1,4 sequence)

$$\sum_{k=1}^{5} k^{(2)}T_k = 1*0*1 + 2*1*4 + 3*2*5 + 4*3*9 + 5*4*14 = 426 .$$

By formula

$$5*4*37 - 2*4*60 + 2*97 - 2*9 - 2*5 = 426 .$$

THE SUMMATION

$$\sum_{k=1}^{n} k^{(3)}T_k = \varphi(n)$$

$$\Delta\varphi(n) = (n+1)^{(3)}T_{n+1}$$

$$\Delta n^{(3)}T_{n+2} = (n+1)^{(3)}T_{n+1} + 3n^{(2)}T_{n+2}$$

$$\sum_{k=1}^{n} k^{(3)}T_k = n^{(3)}T_{n+2} - 3(n-1)^{(2)}T_{n+3} + 6(n-2)T_{n+4} - 6T_{n+5} + 6T_6 + C .$$

For $n = 3$,

$$6T_3 = 6T_5 - 6T_6 + 6T_7 - 6T_8 + 7T_6 + C; \qquad C = 6T_5$$

$$\sum_{k=1}^{n} k^{(3)}T_k = n^{(3)}T_{n+2} - 3(n-1)^{(2)}T_{n+3} + 6(n-2)T_{n+4} - 6T_{n+5} + 6T_7 .$$

VERIFICATION (1,4 sequence)

$$\sum_{k=1}^{6} k^{(3)}T_k = 6*5 + 24*9 + 60*14 + 120*23 = 3846.$$

By formula for $n = 6$,

$$120*60 - 60*97 + 24*157 - 6*254 + 6*37 = 3846.$$

The formulas for the next two cases are written down and the pattern that is emerging is noted.

$$\sum_{k=1}^{n} k^{(4)}T_k = n^{(4)}T_{n+2} - 4(n-1)^{(3)}T_{n+3} + 12(n-2)^{(2)}T_{n+4} - 24(n-3)T_{n+5} + 24T_{n+6} - 24T_9 .$$

$$\sum_{k=1}^{n} k^{(5)}T_k = n^{(5)}T_{n+2} - 5(n-1)^{(4)}T_{n+3} + 20(n-2)^{(3)}T_{n+4} - 60(n-3)^{(2)}T_{n+5}$$
$$+ 120(n-4)T_{n+6} - 120T_{n+7} + 120T_{11} .$$

The pattern may be described as follows:

For the r^{th} difference:

1. The first term is $n^{(r)}T_{n+2}$.
2. For the n portion, both n and r go down by 1 at each step.
3. For the T portion the subscript goes up by 1 at each step for $r+1$ steps.
4. The signs alternate.
5. The coefficients are the product, respectively, of the binomial coefficients for r by 0!, 1!, 2!, $\cdots$, $r!$, respectively.
6. The last term is $r!T_{2r+1}$ with sign determined by the alternation mentioned in 4.

With the aid of these factorial formulas it is now possible to find polynomial formulas. For example.

$$\sum_{k=1}^{n} k^4T_k = \sum_{k=1}^{n} [k^{(4)} + 6k^{(3)} + 7k^{(2)} + k^{(1)}]T_k .$$

The first few formulas for the powers are given herewith.

$$\sum_{k=1}^{n} k^2T_k = (n^2+2)T_{n+2} - (2n-3)T_{n+3} - T_6$$

$$\sum_{k=1}^{n} k^3T_k = (n^3+6n-12)T_{n+2} - (3n^2-9n+19)T_{n+3} + 6T_6 + T_3$$

$$\sum_{k=1}^{n} k^4T_k = (n^4+12n^2-48n+98)T_{n+2} - (4n^3-18n^2+76n-159)T_{n+3} - 13T_8 - 11T_7$$

$$\sum_{k=1}^{n} k^5T_k = (n^5+20n^3-120n^2+490n-1020)T_{n+2} - (5n^4-30n^3+190n^2-795n+1651)T_{n+3}$$
$$+ 120T_9 + 30T_6 + T_3 .$$

In these formulas considerable algebra has been done to reduce the number of terms down to two main terms by using Fibonacci shift formulas.

GENERAL SECOND-ORDER RECURSION SEQUENCES

Given a second-order recursion sequence governed by the recursion relation

$$T_{n+1} = P_1T_n + P_2T_{n-1}$$

to find

$$\sum_{k=1}^{n} T_k = \varphi(n)$$

$$\Delta\varphi(n) = T_{n+1}$$

$$\Delta[T_n + P_2T_{n-1}] = T_{n+1} + P_2T_n - T_n - P_2T_{n-1} = (P_1 + P_2 - 1)T_n.$$

Provided $P_1 + P_2 - 1$ is not zero,

$$\sum_{k=1}^{n} T_k = (T_{n+1} + P_2T_n)/(P_2 + P_1 - 1) + C.$$

For $n = 1$,

$$T_1 = (T_2 + P_2T_1)/(P_2 + P_1 - 1) + C$$

$$C = [(P_1 - 1)T_1 - T_2]/(P_2 + P_1 - 1)$$

$$\sum_{k=1}^{n} T_k = [T_{n+1} + P_2T_n + (P_1 - 1)T_1 - T_2]/(P_2 + P_1 - 1).$$

EXAMPLE:

$$T_{n+1} = 5T_n - 3T_{n-1}$$

3, 7, 26, 109, 467, 2008;

$$\sum_{k=1}^{5} T_k = 3 + 7 + 26 + 109 + 467 = 612.$$

By formula (2008 – 3*467 + 4*3 – 7)/(5 – 3 – 1) = 612.

SUM OF TERMS OF A THIRD-ORDER SEQUENCE

Such a sequence is bound by a recursion relation of the form

$$T_{n+1} = P_1T_n + P_2T_{n-1} + P_3T_{n-2}.$$

If

$$\sum_{k=1}^{n} T_k = \varphi(n), \qquad \Delta\phi(n) = T_{n+1}$$

$$\Delta(T_n + (P_3 + P_2)T_{n-1} + P_3T_{n-2}) = T_{n+1} + (P_3 + P_2)T_n + P_3T_{n-1} - T_n - (P_3 + P_2)T_{n-1} - P_3T_{n-2}$$
$$= T_{n+1} + (P_3 + P_2 - 1)T_n - P_2T_{n-1} - P_3T_{n-2} = (P_1 + P_2 + P_3 - 1)T_n.$$

Hence if $P_1 + P_2 + P_3 - 1$ is not zero,

$$\sum_{k=1}^{n} T_k = [T_{n+1} + (P_3 + P_2)T_n + P_3T_{n-1}]/(P_1 + P_2 + P_3 - 1) + C$$

$$T_1 + T_2 = [T_3 + (P_3 + P_2)T_2 + P_3T_1]/(P_1 + P_2 + P_3 - 1) + C$$

$$C = [(P_1 + P_2 - 1)T_1 + (P_1 - 1)T_2 - T_3]/(P_1 + P_2 + P_3 - 1)$$

$$\sum_{k=1}^{n} T_k = [T_{n+1} + (P_3 + P_2)T_n + P_3T_{n-1} + (P_1 + P_2 - 1)T_1 + (P_1 - 1)T_2 - T_3]/(P_1 + P_2 + P_3 - 1)$$

EXAMPLE.

$$T_{n+1} = 3T_n + 2T_{n-1} - T_{n-2}$$

1 + 2 + 4 + 15 + 179 = 252. Next term is 624.

By formula (624 + 179 – 51 + 4*1 + 2*2 – 4)/3 = 252.

FOURTH-ORDER SEQUENCES

The recursion relation is

$$T_{n+1} = P_1 T_n + P_2 T_{n-1} + P_3 T_{n-2} + P_4 T_{n-3}.$$

An entirely similar analysis as was made for third-order sequences leads to the formula

$$T_k = [T_{n+1} + (P_2 + P_3 + P_4)T_n + (P_3 + P_4)T_{n-1} + P_4 T_{n-2}]/(P_1 + P_2 + P_3 + P_4 - 1) + C,$$

where

$$C = [(P_1 + P_2 + P_3 - 1)T_1 + (P_1 + P_2 - 1)T_2 + (P_1 - 1)T_3 - T_4]/(\Sigma P_i - 1).$$

EXAMPLE. $T_{n+1} = 3T_n + 2T_{n-1} - 4T_{n-2} + 3T_{n-3}$

1 + 3 + 4 + 6 + 17 + 56 + 190 + 632 = 909. Next term is 2103. By formula (2103 + 632 − 190 + 3*56 + 4*3 + 2*4 − 6)/3 = 909.

FIBONACCI-COMBINATORIAL FORMULAS

These are closely related to the Fibonacci-factorial formulas discussed on pp. 13–15. However the added simplicity of these formulas merits a listing of the first few to show the pattern.

$$\sum_{k=1}^{n} \binom{k}{1} T_k = \binom{n}{1} T_{n+2} - T_{n+3} + T_3, \qquad \sum_{k=1}^{n} \binom{k}{2} T_k = \binom{n}{2} T_{n+2} - \binom{n-1}{1} T_{n+3} + T_{n+4} - T_5$$

$$\sum_{k=1}^{n} \binom{k}{3} T_k = \binom{n}{3} T_{n+2} - \binom{n-1}{2} T_{n+3} + \binom{n-2}{1} T_{n+4} - T_{n+5} + T_7$$

$$\sum_{k=1}^{n} \binom{k}{4} T_k = \binom{n}{4} T_{n+2} - \binom{n-1}{3} T_{n+3} + \binom{n-2}{2} T_{n+4} - \binom{n-3}{1} T_{n+5} + T_{n+6} - T_9$$

$$\sum_{k=1}^{n} \binom{k}{5} T_k = \binom{n}{5} T_{n+2} - \binom{n-1}{4} T_{n+3} + \binom{n-2}{3} T_{n+4} - \binom{n-3}{2} T_{n+5} + \binom{n-4}{1} T_{n+6} - T_{n+7} + T_{11}$$

FIBONACCI EXTENSION: SUMMING MORE TERMS

Sequences governed by

$$T_{n+1} = T_n + T_{n-1} + T_{n-2},$$

where three rather than two preceding terms are added at each step have a summation formula

$$\sum_{k=1}^{n} T_k = (T_{n+1} + 2T_n + T_{n-1} + T_1 - T_3)/2.$$

For sequences governed by

$$T_{n+1} = T_n + T_{n-1} + T_{n-2} + T_{n-3},$$

where the four previous terms are added

$$\sum_{k=1}^{n} T_k = (T_{n+1} + 3T_n + 2T_{n-1} + T_{n-2} + 2T_1 + T_2 - T_4)/3.$$

Where five previous terms are added at each step:

$$\sum_{k=1}^{n} T_k = (T_{n+1} + 4T_n + 3T_{n-1} + 2T_{n-2} + T_{n-3} + 3T_1 + 2T_2 + T_3 - T_5)/4.$$

Where six previous terms have been added at each step:

$$\sum_{k=1}^{n} T_k = (T_{n+1} + 5T_n + 4T_{n-1} + 3T_{n-2} + 2T_{n-3} + T_{n-4} + 4T_1 + 3T_2 + 2T_3 + T_4 - T_6)/5.$$

EXAMPLE.

$$1 + 2 + 4 + 5 + 7 + 8 + 27 + 53 + 104 + 204 = 415 .$$

By formula

$$(403 + 5*204 + 4*104 + 3*53 + 2*27 + 8 + 4 + 6 + 8 + 5 - 8)/5 = 415.$$

CONCLUSION

Finite differences have wide application in formula development. There are, of course, many situations in which the use of this method leads to difficulties which other procedures can obviate. But where applicable the results are often obtained with such facility that other procedures seem laborious by comparison.

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A GOLDEN DOUBLE CROSTIC

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Use the definitions in the clue story which follows to write the words to which they refer; then enter the appropriate letters in the diagram to complete a quotation from a mathematician whose name appears in the last line of the diagram. The name of the book in which this quotation appeared and the author's last name appear as the first letters of the clue words. The end of each word is indicated by a shaded square following it.

CLUE STORY

The mystic Golden Section Ratio, $(1 + \sqrt{5})/2$, called __(A-1, A-2)__ (the latter most commonly), occurs in several propositions in __(A-3, A-4)__ on line segments and __(A-5)__. This Golden Cut fascinated the ancient Greeks, particularly the __(D-1)__, who found this value in the ratio of lengths of segments in the __(D-2)__ and __(D-3)__ and who also made studies in __(D-4)__. The Greeks found the proportions of the Golden Rectangle most pleasing to the eye as evidenced by the ubiquitous occurrence of this form in art and architecture, such as __(C-1)__ or in sculpture as in the proportions of the famous __(C-2)__; however, they may have been copying __(C-3)__, for the Golden Proportion occurs frequently in the forms of living things and is closely related to the growth patterns of plants, as __(C-4, C-5, C-6)__, in which occur ratios of Fibonacci numbers. The Golden Section is the limiting value of the ratio of two successive Fibonacci numbers (named for __(G-1)__), being closely approximated by the __(G-2, G-3)__.

By some mathematicians, the beauty of the __(N)__ relating to the Golden Section is compared to the theorem of the __(D-1)__ and to such results from projective geometry as those seen in Pascal's "Mystic __(B)__" or even in the applications of mathematics in the *Principia Mathematica* of __(I)__ while the constant $(1 + \sqrt{5})/2$ itself is rivalled by __(E-1)__ and __(E-2)__.

Unfortunately, not all persons find mathematics beautiful. __(H-1)__ was one of the four branches of arithmetic given by the Mock Turtle in *Alice in Wonderland,* and the card player's description of the sequence 2, 1, 3, 4, 7, ______, 18, 29, 47, ··· would be __(H-2)__, while some have to have all mathematics of practical use, such as in reading an __(M)__.

[The solution appears on page 83 of the *Quarterly.*)

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