STOLARSKY'S DISTRIBUTION OF THE POSITIVE INTEGERS

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Let F_n be the n^{th} Fibonacci number, where $F_1 = 1$, $F_2 = 2$ and $F_{n+2} = F_{n+1} + F_n$, $\forall n \in N$. It is well known that

$$\lim_{n \to \infty} F_{n+1}/F_n = \alpha = \tfrac{1}{2}(1+\sqrt{5}),$$

the larger root of the polynomial equation $x^2 = x + 1$. Using the mapping $g : N \to N$,

$$g(r) = [r\alpha + \tfrac{1}{2}],$$

i.e., $g(r)$ is the closest integer to $r\alpha$, we can give an alternate formulation of F_n. It is easy to show that,

$$g(F_n) = F_{n+1}, \forall n \in N,$$

so as $F_1 = 1$,

$$F_n = g^{n-1}(1), \forall n \in N,$$

where we set

$$g^o(r) = r, \quad \text{and} \quad g^n(r) = g(g^{n-1}(r)), \forall n \in N.$$

Hence the Fibonacci sequence is

$$(F_n) = (g^{n-1}(1)).$$

For each $r \in N$, we will show that the sequence $(g^{n-1}(r))$ has the Fibonacci recursive property

$$g^{n+1}(r) = g^n(r) + g^{n-1}(r), \forall n \in N.$$

K. Stolarsky constructed a table of these sequences to cover the positive integers in the following way. $\forall\, m,n \in N$, we define:

(a) $S(m,1)$ = least positive integer not in $T(m) = \{S(i,j) : j \in N,\ i = 1, \cdots, m-1\}$;

(b) $$S(m,n+1) = g(S(m,n)).$$

Effectively what is being constructed is a table of sequences $g^{n-1}(r)$, where r is least integer not in an earlier sequence and, $r = 1$ is the starting value for the first sequence, the Fibonacci sequence. Obviously, by construction S will cover N.

In Table 1, we list the 100 values of $S(m,n)$ for $m,n \leqslant 10$. It is easily shown (Theorem 1), that each positive integer r occurs exactly once as a value $S(m,n)$, and that $S(m,n+2) - S(m,n+1) = S(m,n)$, (Lemma 1).

Table 1

	n = 1	2	3	4	5	6	7	8	9	10
m = 1	1	2	3	5	8	13	21	34	55	89
2	4	6	10	16	26	42	68	110	178	288
3	7	11	18	29	47	76	123	199	322	521
4	9	15	24	39	63	102	165	267	432	699
5	12	19	31	50	81	131	212	343	555	898
6	14	23	37	60	97	157	254	411	665	1076
7	17	28	45	73	118	190	308	499	808	1307
8	20	32	52	84	136	220	356	576	932	1508
9	22	36	58	94	152	246	398	644	1042	1686
10	25	40	65	105	170	275	445	720	1165	1885

Stolarsky observed in his table, as far as he had calculated, that the differences between the values in columns 2 and 1 of a given row, $S(m,2) - S(m,1)$, were always integers that had previously occurred in one of these two columns. He conjectured that this was always the case. J. Butcher conjectured further, on the basis of computation, that this correspondence was one-to-one.

In this paper we prove both these conjectures, as well as constructing other interesting properties of $S(m,n)$. To facilitate our construction, we define the following functions:

$$d : N \to N,\ d(m) = S(m,2) - S(m,1)\,;$$

$$h : N \to (-\tfrac{1}{2},\tfrac{1}{2}),\ h(r) = r\alpha - g(r);$$

and

$$k : N \to N,\ k(r) = [1 - \log_\alpha |2h(r)|]\,.$$

Hence $d(m)$ is the difference between columns 2 and 1 in row m, and $h(r)$ is the "closeness" of $r\alpha$ to the nearest integer.

We will show firstly that S is a one-to-one and onto map $N \times N$ to N:

Theorem 1.

$$\forall r \in N,\ \exists 1\, m,n \in N : r = S(m,n)\,.$$

We will use this result to establish Stolarsky's conjecture:

Theorem 2.

$$\forall m \in N, \exists\, n \in N : d(m) = S(n,1) \text{ or } d(m) = S(n,2)\,.$$

We will then improve Theorem 1 by finding explicit invertible formulae relating m,n to $S(m,n)$:

Theorem 3.

$$S(m,1) = [m\alpha^2 - \tfrac{1}{2}\alpha],\ S(m,n) = g^{n-1}(S(m,1)),\ \forall m,n \in N,$$

$$n = k(S(m,n)),\ m = [S(m,n)\alpha^{-n-1} + \tfrac{1}{2}\alpha]$$

Further we note that the sequence m, $d(m)$, $S(m,1)$ can be approximated by m, $m\alpha$ and $m\alpha^2$, or more explicitly:

Theorem 4

For $h(m) \in (-\tfrac{1}{2}, -\tfrac{1}{2}\alpha^{-2})$,

$$d(m) = g(m) - 1, \quad S(m,1) = g(d(m)) - 1\,;$$

for $h(m) \in (-\tfrac{1}{2}\alpha^{-2}, \tfrac{1}{2}\alpha^{-1})$,

$$d(m) = g(m) - 1, \quad S(m,1) = g(d(m)) + 1\,;$$

and for $h(m) \in (\tfrac{1}{2}\alpha^{-1}, \tfrac{1}{2})$,

$$d(m) = g(m), \quad S(m,1) = g(d(m)) - 1\,.$$

This theorem leads to explicit invertible formulae relating $d(m)$ to $S(n,1)$ and $S(n,2)$:

Theorem 5.

For $h(m) \in (-\tfrac{1}{2}, \tfrac{1}{2}\alpha^{-3})$,

$$d(m) = S([m\alpha^{-1} + \tfrac{1}{2}],\ 1)\,;$$

for $h(m) \in (\tfrac{1}{2}\alpha^{-3}, \tfrac{1}{2})$,

$$d(m) = S([m\alpha^{-2} + \tfrac{1}{2}],\ 2)\,;$$

while

$$S(m,1) = d([m\alpha + \tfrac{1}{2}\alpha^{-2}]), \quad \text{and} \quad S(m,2) = d([m\alpha^2 - \tfrac{1}{2}\alpha^{-1}])\,.$$

This leads finally to establishing Butcher's conjecture:

Theorem 6.

$$\{d(m) : m \in N\} = \{S(m,1) : m \in N\} \cup \{S(m,2) : m \in N\}\,.$$

We will now prove these theorems via the following lemmas. We will frequently use identities based on $a^2 = a + 1$, of the form

$$a^{n+1} = F_n a + F_{n-1}, \quad \forall n \in N,$$
$$a^{-n} = (-1)^n (F_n - F_{n-1} a), \quad \forall n \in N.$$

Lemma 1. $\forall r \in N, \quad g^2(r) = g(r) + r.$

Proof.
$$ar - \tfrac{1}{2} < g(r) < ar + \tfrac{1}{2},$$
$$\Rightarrow ag(r) - \tfrac{1}{2} < g^2(r) < ag(r) + \tfrac{1}{2},$$
$$\Rightarrow (a-1)g(r) - \tfrac{1}{2} < g^2(r) - g(r) < (a-1)g(r) + \tfrac{1}{2}.$$

But
$$a(a-1)r - \tfrac{1}{2}(a-1) < (a-1)g(r) < a(a-1)r + \tfrac{1}{2}(a-1),$$
and
$$a(a-1) = a^2 - a = 1,$$
so
$$r - \tfrac{1}{2}(a-1) - \tfrac{1}{2} < g^2(r) - g(r) < r + \tfrac{1}{2}(a-1) + \tfrac{1}{2},$$
$$\Rightarrow r - 1 < r - \tfrac{1}{2}a < g^2(r) - g(r) < r + \tfrac{1}{2}a < r + 1.$$

Hence as $g^2(r) - g(r)$ is integral, $g^2(r) - g(r) = r$, and the result of the lemma follows.

Corollary. $S(1,n) = F_n.$

Proof. $T(1) = \varphi \Rightarrow S(1,1) = 1 = F_1, \quad S(1,2) = g(1) = 2 = F_2.$

By Lemma 1,
$$S(1, n+2) = g^2(S(1,n)) = g(S(1,n)) + S(1,n) = S(1, n+1) + S(1,n), \quad \forall n \in N$$
so by induction,
$$S(1, n+2) = F_{n+1} + F_n = F_{n+2}, \quad \forall n \in N.$$

As we move from left to right across the table we find that each value $g(n)a$ gives a better approximation to an integer $(g^2(n))$ than did na, $(g(n))$. Explicitly we have the following recursive result.

Lemma 2. $\forall n \in N, \quad h(g(n)) = -a^{-1}h(n).$

Proof.
$$\begin{aligned} h(g(n)) &= ag(n) - g^2(n), \\ &\equiv ag(n) \pmod 1, \\ &\equiv a^2 n - ah(n) \pmod 1, \\ &\equiv an - ah(n) \pmod 1, \quad (\text{as } a^2 = a + 1), \\ &\equiv h(n) - ah(n) \pmod 1, \\ &\equiv (1-a)h(n) \pmod 1. \end{aligned}$$
$$1 - a = -a^{-1}, \quad |hg(n)| < \tfrac{1}{2}, \quad |-a^{-1}h(n)| < \tfrac{1}{2},$$

so $h(g(n)) = -a^{-1}h(n)$.

Lemma 2 enables us to prove the following relation between $S(m,n)$ and n, namely that r occurs in the $k(r)^{th}$ column of the table.

Lemma 3. $k(S(m,n)) = n, \quad \forall m, n \in N.$

Proof. Let $r = [S(m,1)a^{-1}]$, and set $\epsilon = ra - S(m,1)$. $0 < \epsilon < 1$.

For $m > 1$, $S(m,1) - 2 < g(r) < S(m,1) + 1$, so
$$g(r) = S(m,1) \quad \text{or} \quad S(m,1) - 1.$$

But $g(r) \in T(m)$ as $r < S(m,1)$, and $S(m,1) \notin T(m)$ so
$$g(r) = S(m,1) - 1, \quad \forall m > 1.$$

Also $g(0) = [\tfrac{1}{2}] = 0$, $S(1,1) - 1 = 0$, so

$$g(r) = S(m,1) - 1, \quad \forall\, m \in N.$$

Hence

$$\begin{aligned} S(m,1) &= [ar + \tfrac{1}{2}] + 1, \\ &= [S(m,1) + \tfrac{1}{2} - \epsilon] + 1 \\ &\Rightarrow \epsilon > \tfrac{1}{2}. \end{aligned}$$

Further,

$$\begin{aligned} h(S(m,1)) &\equiv aS(m,1) \pmod 1, \\ &\equiv a^{-1}S(m,1) \pmod 1, \quad (\text{as } a = 1 + a^{-1}), \\ &\equiv -\epsilon a^{-1} \pmod 1. \end{aligned}$$

Hence, for $\epsilon < \tfrac{1}{2}a$,

$$h(S(m,1)) = -\epsilon a^{-1} < -\tfrac{1}{2}a^{-1},$$

and for $\epsilon > \tfrac{1}{2}a$,

$$h(S(m,1)) = 1 - \epsilon a^{-1} > 1 - a^{-1} > \tfrac{1}{2}a^{-1},$$

Thus in both cases

$$|h(S(m,1))| \in (\tfrac{1}{2}a^{-1}, \tfrac{1}{2}) \Rightarrow k(S(m,1)) = 1.$$

Now using Lemma 2, $k(S(m,n+1)) = k(S(m,n)) + 1$, so by induction, $k(S(m,n)) = n$.

This means an integer r cannot appear in two different columns. In the next lemma, we show that no integer can appear more than once in any given column.

Lemma 4. $$S(m+1, n) > S(m,n), \quad \forall\, m,n \in N.$$

Proof. By definition $S(m,1)$ is not the least integer in $T(m)$, and $S(m+1, 1)$ the least integer not in $T(m+1) \supseteq T(m) \cup \{S(m,1)\}$, so $S(m+1, 1) \geqslant S(m,1) + 1$. Also

$$\begin{aligned} S(m+1, 2) &= g(S(m+1, 1)), \\ &\geqslant aS(m+1, 1) - \tfrac{1}{2}, \\ &> a(S(m,1) + 1) - \tfrac{1}{2}, \\ &> aS(m,1) + \tfrac{1}{2}, \\ &\geqslant g(S(m,1)), \\ &= S(m,2), \end{aligned}$$

i.e., $S(m+1, 2) > S(m,2)$. Now by induction, using Lemma 1,

$$\begin{aligned} S(m+1, n+2) &= S(m+1, n+1) + S(m+1, n) > S(m, n+1) + S(m,n) \\ &= S(m, n+2). \qquad \forall m, n \in N. \end{aligned}$$

Combining this final result with the two initial results we prove the lemma.

Lemmas 3 and 4 now enable us to prove Theorem 1. By the sieve type definition $S : N \times N \to N$ must be onto. If $S(m_1, n_1) = S(m_2, n_2) = r$ say, then by Lemma 3, $n_1 = n_2 = k(r)$ and then by Lemma 4 $m_1 = m_2$. Hence S is one-to-one. We have proved:

Theorem 1. $$\forall\, r \in N, \ \exists\, 1 m, n \in N : r = S(m,n).$$

Stolarsky's conjecture can now be established by proving one more Lemma.

Lemma 5. $$k(d(m)) \leqslant 2, \quad \forall\, m \in N.$$

Proof.

$$\begin{aligned} h(d(m)) &\equiv ad(m) \pmod 1, \\ &\equiv aS(m,2) - aS(m,1) \pmod 1, \\ &\equiv h(S(m,2)) - h(S(m,1)) \pmod 1, \end{aligned}$$

Now by Lemma 2, $h(S(m,2)) = -a^{-1}h(S(m,1))$, so

$$h(d(m)) \equiv -(1+a^{-1})h(S(m,1)) \pmod 1,$$
$$\equiv -ah(S(m,1)) \pmod 1.$$

Further $h(S(m,1)) \in (-\tfrac12, -\tfrac12 a^{-1}) \cup (\tfrac12 a^{-1}, \tfrac12)$ by Lemma 3, so

$$h(d(m)) = 1 - ah(S(m,1)) \quad \text{if} \quad h(S(m,1)) \in (-\tfrac12, -\tfrac12 a^{-1})$$

and

$$h(d(m)) = -1 - ah(S(m,1)) \text{ if } h(S(m,1)) \in (\tfrac12 a^{-1}, \tfrac12),$$

so in either case

$$|h(d(m))| = 1 - a|h(S(m,1))|,$$
$$> 1 - \tfrac12 a,$$
$$= \tfrac12 a^{-2}.$$

Hence $k(d(m)) \leqslant 2$.

As by Theorem 1, the value $r = d(m)$ can occur in only one position, and as $k(d(m)) \leqslant 2$, by Lemma 3, $d(m)$ appears in Column 1 or Column 2. Hence we have established our second theorem.

Theorem 2. $\forall m \in N, \exists n \in N : d(m) = S(n,1)$ or $d(m) = S(n,2)$.

We now return to improve the result of Theorem 1 by finding an explicit relationship between m,n and $S(m,n)$. We note first

Lemma 6. $k([na^2 - \tfrac12 a]) = 1$.

Proof. Let $r = [na^2 - \tfrac12 a]$. Now

$$na^2 - \tfrac12 a \equiv na - \tfrac12 a \pmod 1,$$
$$\equiv h(n) - \tfrac12 a \pmod 1.$$

Also $-2 < -\tfrac12 - \tfrac12 a < h(n) - \tfrac12 a < \tfrac12 - \tfrac12 a < 0$, so

$$r = na^2 - h(n) - t,$$

where

$$t = 2 \quad \text{for} \quad -\tfrac12 < h(n) < \tfrac12 a - 1 = -\tfrac12 a^{-2},$$

and

$$t = 1 \quad \text{for} \quad -\tfrac12 a^{-2} < h(n) < \tfrac12.$$

Further

$$h(r) \equiv ra \pmod 1,$$
$$\equiv na^3 - h(n)a - ta \pmod 1,$$
$$\equiv 2na - h(n)a - ta \pmod 1,$$
$$\equiv h(n)(2-a) - ta \pmod 1,$$
$$\equiv h(n)a^{-2} - ta \pmod 1.$$

For $-\tfrac12 < h(n) < -\tfrac12 a^{-2}$,

$$t = 2 \Rightarrow -ta \equiv -2a \equiv -a^3 \pmod 1,$$
$$\Rightarrow -\tfrac12 < -\tfrac12 a^{-2} - a^{-3} < h(n)a^{-2} - a^{-3} < -\tfrac12 a^{-4} - a^{-3} = -\tfrac12 a^{-1},$$
$$\Rightarrow h(r) = h(n)a^{-2} - a^{-3} \quad \text{and} \quad k(r) = 1.$$

For $-\tfrac12 a^{-2} < h(n) < \tfrac12 a^{-1}$,

$$t = 1 \Rightarrow -ta \equiv -a \equiv a^{-2} \pmod 1,$$
$$\Rightarrow \tfrac12 a^{-1} < a^{-2}(h(n)+1) < \tfrac12,$$
$$\Rightarrow h(r) = a^{-2}(h(n)+1) \quad \text{and} \quad k(r) = 1.$$

For $\tfrac12 a^{-1} < h(n) < \tfrac12$,

$$\begin{aligned} t = 1 \Rightarrow -t\alpha &\equiv -\alpha \equiv -\alpha^{-1} \pmod 1, \\ \Rightarrow -\tfrac{1}{2} &= \tfrac{1}{2}\alpha^{-3} - \alpha^{-1} \le h(n) < \tfrac{1}{2}\alpha^{-2} - \alpha^{-1} < -\tfrac{1}{2}\alpha^{-1}, \\ \Rightarrow h(r) &= \alpha^{-2}h(n) - \alpha^{-1} \quad \text{and} \quad k(r) = 1. \end{aligned}$$

We can now show that the numbers $[n\alpha^2 - \tfrac{1}{2}\alpha]$ are the only integers occurring in Column 1.

Lemma 7. $$S(n,1) = [n\alpha^2 - \tfrac{1}{2}\alpha].$$

Proof. Let $r = S(n,1)$, then

$$\begin{aligned} h(r+1) &\equiv \alpha(r+1) \pmod 1, \\ &\equiv h(r) + \alpha \pmod 1. \end{aligned}$$

Noting $\alpha \equiv \alpha^{-1} \equiv -\alpha^{-2} \pmod 1$ we find: for $-\tfrac{1}{2} < h(r) < -\tfrac{1}{2}\alpha^{-1}$,

$$\tfrac{1}{2}\alpha^{-3} < h(r) + \alpha^{-1} < \tfrac{1}{2}\alpha^{-1} \Rightarrow k(r+1) > 1;$$

and for $-\tfrac{1}{2}\alpha^{-1} < h(r) < \tfrac{1}{2}$,

$$-\tfrac{1}{2}\alpha^{-4} < h(r) - \alpha^{-2} < \tfrac{1}{2}\alpha^{-3} \Rightarrow k(r+1) > 1.$$

Hence $r + 1$ cannot be in Column 1, so Column 1 cannot contain two consecutive integers.

Let $t(n) = [n\alpha^2 - \tfrac{1}{2}\alpha]$, then

$$n\alpha^2 - \tfrac{1}{2}\alpha - 1 < t(n) < n\alpha^2 - \tfrac{1}{2}\alpha,$$

$$n\alpha^2 + \alpha^2 - \tfrac{1}{2}\alpha - 1 < t(n+1) < n\alpha^2 + \alpha^2 - \tfrac{1}{2}\alpha,$$

so

$$n\alpha^2 + \alpha^2 - \tfrac{1}{2}\alpha - 1 - (n\alpha^2 - \tfrac{1}{2}\alpha) < t(n+1) - t(n) < n\alpha^2 + \alpha^2 - \tfrac{1}{2}\alpha - (n\alpha^2 - \tfrac{1}{2}\alpha - 1),$$

and as

$$n\alpha^2 + \alpha^2 - \tfrac{1}{2}\alpha - 1 - (n\alpha^2 - \tfrac{1}{2}\alpha) = \alpha^2 - 1 = \alpha > 1,$$

and

$$n\alpha^2 + \alpha^2 - \tfrac{1}{2}\alpha - (n\alpha^2 - \tfrac{1}{2}\alpha - 1) = \alpha^2 + 1 = \alpha + 2 < 4,$$

we have

$$1 < t(n+1) - t(n) < 4.$$

Hence $t(n)$ and $t(n+1)$ are distinct integers whose difference is 2 or 3. They both occur in Column 1 (Lemmas 6 and 3), so no other integer can occur in Column 1, as that would imply consecutive integers in Column 1.

We can now specify $S(m,n)$ with the following two lemmas.

Lemma 8. $S(m,n) = S(m,1)\alpha^{n-1} + F_{n-2}h(S(m,1)),\ \forall\, n \in N.$ (Putting $F_0 = 1,\ F_{-1} = 0.$)

Proof. Trivial for $n = 1$.

Assume $S(m,n) = S(m,1)\alpha^{n-1} + F_{n-2}h(S(m,1))$, for some $n > 1$, then

$$\begin{aligned} S(m, n+1) &= g(S(m,n)), \\ &= \alpha S(m,n) + h(S(m,n)), \\ &= S(m,1)\alpha^n + F_{n-2}h(S(m,1))\alpha + h(S(m,n)). \end{aligned}$$

But, by Lemma 2,

$$\begin{aligned} h(S(m,n)) &= -\alpha^{-1}h(S(m,n-1)), \\ &= (-\alpha^{-1})^{n-1}h(S(m,1)), \end{aligned}$$

and as $\alpha^{-(n-1)} = (-1)^{n-1}(F_{n-1} - F_{n-2}\alpha)$,

$$F_{n-2}\alpha + (-\alpha)^{-(n-1)} = F_{n-1}.$$

Hence

$$S(m, n+1) = S(m,1)\alpha^n + F_{n-1}h(S(m,1)).$$

Thus, by induction, this result is true $\forall\, n \in N$.

From this result follows

Lemma 9. $$m = [S(m,n)\alpha^{-n-1} + \tfrac{1}{2}\alpha].$$

Proof. By Lemma 8,

$$|S(m,n) - S(m,1)a^{n-1}| = F_{n-2}|h(S(m,1))| < \tfrac{1}{2}F_{n-2}.$$

Also, $F_{n-2} < a^{n-2}$, so

$$|S(m,n) - S(m,1)a^{n-1}| < \tfrac{1}{2}a^{-3}.$$

From Lemma 7

$$ma^2 - \tfrac{1}{2}a - 1 < S(m,1) < ma^2 - \tfrac{1}{2}a,$$
$$\Rightarrow -\tfrac{1}{2}a^{-1} - a^{-2} < S(m,1)a^{-2} - m < -\tfrac{1}{2}a^{-1}.$$

But, from above,

$$-\tfrac{1}{2}a^{-3} < S(m,n)a^{-n-1} - S(m,1)a^{-2} < \tfrac{1}{2}a^{-3},$$

so adding

$$-\tfrac{1}{2}a = -\tfrac{1}{2}a^{-1} - a^{-2} - \tfrac{1}{2}a^{-3} < S(m,n)a^{-n-1} - m < \tfrac{1}{2}a^{-3} - \tfrac{1}{2}a^{-1}$$
$$\Rightarrow 0 < S(m,n)a^{-n-1} - m + \tfrac{1}{2}a < \tfrac{1}{2}a + \tfrac{1}{2}a^{-3} - \tfrac{1}{2}a^{-1} = a^{-1} < 1,$$
$$\Rightarrow m = [S(m,n)a^{-n-1} + \tfrac{1}{2}a].$$

This lemma concludes the results for Theorem 3, so combining the results of Lemmas 3, 7 and 9 we have:

Theorem 3. $S(m,1) = [ma^2 - \tfrac{1}{2}a], \quad S(m,n) = g^{n-1}(S(m,1)), \quad \forall\, m,n \in N,$

$$n = k(S(m,n)), \quad m = [S(m,n)a^{-n-1} + \tfrac{1}{2}a].$$

We now examine formulae for $d(m)$.

Lemma 10. $$d(m) = [ma - \tfrac{1}{2}a^{-1}].$$

Proof. Let

$$c(m) = [ma - \tfrac{1}{2}a^{-1}],$$

and set

$$\gamma = ma - \tfrac{1}{2}a^{-1} - c(m), \quad 0 < \gamma < 1.$$

As $S(m,1) = [ma^2 - \tfrac{1}{2}a]$, let

$$\epsilon = ma^2 - \tfrac{1}{2}a - S(m,1), \quad 0 < \epsilon < 1.$$

Now

$$\begin{aligned}\epsilon - \gamma &= m(a^2 - a) + \tfrac{1}{2}(a^{-1} - a) + c(m) - S(m,1),\\ &= m - \tfrac{1}{2} + c(m) - S(m,1),\\ &\equiv \tfrac{1}{2} \pmod 1.\end{aligned}$$

Thus for $\epsilon < \tfrac{1}{2}$, $\gamma = \epsilon + \tfrac{1}{2}$,

$$S(m,1) = c(m) + m,$$

and for $\epsilon > \tfrac{1}{2}$, $\gamma = \epsilon - \tfrac{1}{2}$,

$$S(m,1) = c(m) + m - 1.$$

Further

$$\begin{aligned}c(m) + S(m,1) &= m(a^2 + a) - \tfrac{1}{2}(a + a^{-1}) - (\epsilon + \gamma),\\ &= ma^3 - \tfrac{1}{2}(a^3 - 2) - (\epsilon + \gamma),\\ &= (m - \tfrac{1}{2})a^3 + (1 - \epsilon - \gamma),\end{aligned}$$

and

$$\begin{aligned}S(m,2) &= g(S(m,1)),\\ &= aS(m,1) - h(S(m,1)),\\ &= ma^3 - \tfrac{1}{2}a^2 - \epsilon a - h(S(m,1)).\end{aligned}$$

Combining these two results we find

$$\begin{aligned}c(m) + S(m,1) - S(m,2) &= \tfrac{1}{2}(a^2 - a^3) + (\epsilon a - \epsilon - \gamma) - h(S(m,1)) + 1,\\ &= 1 - \tfrac{1}{2}a + (\epsilon a - \epsilon - \gamma) - h(S(m,1)).\end{aligned}$$

For $0 < \epsilon < \tfrac{1}{2}$, $\gamma = \epsilon + \tfrac{1}{2}$,

$$c(m) + S(m,1) - S(m,2) = 1 - \tfrac{1}{2}a + \epsilon(a-2) - \tfrac{1}{2} - h(S(m,1)) \in (-1, 1 - \tfrac{1}{2}a),$$

and is integral, so

$$c(m) = S(m,2) - S(m,1) = d(m).$$

For $\tfrac{1}{2} < \epsilon < 1$,

$$\gamma = \epsilon - \tfrac{1}{2},$$

$$c(m) + S(m,1) - S(m,2) = 1 - \tfrac{1}{2}a + \epsilon(a-2) + \tfrac{1}{2} - h(S(m,1)) \in (\tfrac{1}{2}a - 1, 1),$$

and is integral, so

$$c(m) = S(m,2) - S(m,1) = d(m).$$

We can now formulate the relationship between m and $d(m)$.

Lemma 11. For $h(m) \in (-\tfrac{1}{2}, \tfrac{1}{2}a^{-1})$,

$$d(m) = g(m) - 1,$$

for $h(m) \in (\tfrac{1}{2}a^{-1}, \tfrac{1}{2})$

$$d(m) = g(m).$$

Proof. For $h(m) \in (-\tfrac{1}{2}, \tfrac{1}{2}a^{-1})$,

$$g(m) = ma - h(m) \in (ma - \tfrac{1}{2}a^{-1}, ma + \tfrac{1}{2}).$$

Now this interval has length $\tfrac{1}{2}a^{-1} + \tfrac{1}{2} = \tfrac{1}{2}a < 1$, and $g(m)$ is integral, so

$$g(m) = [ma - \tfrac{1}{2}a^{-1}] + 1 = d(m) + 1,$$

by Lemma 10.

For $h(m) \in (\tfrac{1}{2}a^{-1}, \tfrac{1}{2})$,

$$g(m) = ma - h(m) \in (ma - \tfrac{1}{2}, ma - \tfrac{1}{2}a^{-1}).$$

This interval has length $\tfrac{1}{2} - \tfrac{1}{2}a^{-1} = 1 - \tfrac{1}{2}a < 1$, so

$$g(m) = [ma - \tfrac{1}{2}a^{-1}] = d(m).$$

Lemma 12. For $h(m) \in (-\tfrac{1}{2}, -\tfrac{1}{2}a^{-2})$,

$$h(d(m)) = -a^{-1}(h(m)+1), \quad k(d(m)) = 1,$$

for $h(m) \in (-\tfrac{1}{2}a^{-2}, \tfrac{1}{2}a^{-3})$,

$$h(d(m)) = 1 - a^{-1}(h(m)+1), \quad k(d(m)) = 1,$$

for $h(m) \in (\tfrac{1}{2}a^{-3}, \tfrac{1}{2}a^{-1})$,

$$h(d(m)) = 1 - a^{-1}(h(m)+1), \quad k(d(m)) = 2,$$

for $h(m) \in (\tfrac{1}{2}a^{-1}, \tfrac{1}{2})$,

$$h(d(m)) = -a^{-1}(h(m)), \quad k(d(m)) = 2.$$

Proof. From Lemma 11

$$h(d(m)) = h(g(m) - \ell),$$

where $\ell = 0$ if $h(m) > \tfrac{1}{2}a^{-1}$, $\ell = 1$ otherwise. Hence

$$\begin{aligned} h(d(m)) &\equiv ag(m) - a\ell \pmod 1, \\ &\equiv ma^2 - ah(m) - a\ell \pmod 1, \\ &\equiv ma - ah(m) - a\ell \pmod 1, \\ &\equiv h(m)(1-a) - a\ell \pmod 1, \\ &\equiv -a^{-1}(h(m)+\ell) \pmod 1. \end{aligned}$$

For $h(m) \in (-\tfrac{1}{2}, -\tfrac{1}{2}a^{-2})$, $\ell = 1$,

$$\Rightarrow -\tfrac{1}{2} = -a^{-1}(1 - \tfrac{1}{2}a^{-2}) < -a^{-1}(h(m)+1) < -\tfrac{1}{2}a^{-1},$$

$$\Rightarrow h(d(m)) = -a^{-1}(h(m)+1), \qquad k(d(m)) = 1.$$

For $h(m) \in (-\tfrac{1}{2}a^{-2}, \tfrac{1}{2}a^{-1})$, $\ell = 1$,

$$\Rightarrow -\tfrac{1}{2}a = -a^{-1}(1 + \tfrac{1}{2}a^{-1}) < -a^{-1}(h(m)+1) < -\tfrac{1}{2},$$

$$\Rightarrow h(d(m)) = 1 - a^{-1}(h(m)+1).$$

In particular, if $h(m) \in (-\tfrac{1}{2}a^{-2}, \tfrac{1}{2}a^{-3})$,

$$\tfrac{1}{2}a^{-1} = 1 - a^{-1}(\tfrac{1}{2}a^{-3} + 1) < h(d(m)) < 1 - a^{-1}(-\tfrac{1}{2}a^{-2} + 1)) = \tfrac{1}{2} \Rightarrow k(d(m)) = 1,$$

and if $h(m) \in (\tfrac{1}{2}a^{-3}, \tfrac{1}{2}a^{-1})$,

$$\tfrac{1}{2}a^{-1} = 1 - a^{-1}(\tfrac{1}{2}a^{-1} + 1) < h(d(m)) < 1 - a^{-1}(\tfrac{1}{2}a^{-3} + 1) = \tfrac{1}{2}a^{-1} \Rightarrow k(d(m)) = 2.$$

For $h(m) \in (\tfrac{1}{2}a^{-1}, \tfrac{1}{2})$, $\ell = 0$,

$$-\tfrac{1}{2}a^{-1} < -a^{-1}h(m) < -\tfrac{1}{2}a^{-2},$$
$$\Rightarrow h(d(m)) = -a^{-1}h(m), \quad k(d(m)) = 2.$$

Now we can establish the relationship between $d(m)$ and $S(m,1)$.

Lemma 13. For $h(m) \in (-\tfrac{1}{2}, -\tfrac{1}{2}a^{-2}) \cup (\tfrac{1}{2}a^{-1}, \tfrac{1}{2})$,

$$S(m,1) = g(d(m)) - 1.$$

For $h(m) \in (-\tfrac{1}{2}a^{-2}, \tfrac{1}{2}a^{-1})$

$$S(m,1) = g(d(m)) + 1.$$

Proof. For $h(m) \in (-\tfrac{1}{2}, -\tfrac{1}{2}a^{-2})$,

$$\begin{aligned} g(d(m)) &= ad(m) - h(d(m)),\\ &= ag(m) - a + a^{-1}(h(m) + 1), \quad \text{(Lemmas 11 and 12)},\\ &= ma^2 - ah(m) - a + a^{-1}(h(m) + 1),\\ &= ma^2 + (a^{-1} - a)(h(m) + 1),\\ &= ma^2 - (h(m) + 1),\end{aligned}$$
$$\Rightarrow ma^2 - \tfrac{1}{2}a = ma^2 - (1 - \tfrac{1}{2}a^{-2}) < g(d(m)) < ma^2 - \tfrac{1}{2},$$
$$\Rightarrow g(d(m)) = [ma^2 - \tfrac{1}{2}a] + 1 = S(m,1) + 1.$$

For $h(m) \in (-\tfrac{1}{2}a^{-2}, \tfrac{1}{2}a^{-1})$,

$$\begin{aligned} g(d(m)) &= ad(m) - h(d(m)),\\ &= ag(m) - a + a^{-1}(h(m) + 1) - 1, \quad \text{(Lemmas 11 and 12)},\\ &= ma^2 - h(m) - 2,\end{aligned}$$
$$\Rightarrow ma^2 - \tfrac{1}{2}a^{-1} - 2 < g(d(m)) < ma^2 + \tfrac{1}{2}a^{-2} - 2 = ma^2 - \tfrac{1}{2}a - 1,$$
$$\Rightarrow g(d(m)) = [ma^2 - \tfrac{1}{2}a] - 1 = S(m,1) - 1.$$

For $h(m) \in (\tfrac{1}{2}a^{-1}, \tfrac{1}{2})$

$$\begin{aligned} g(d(m)) &= ad(m) - h(d(m)),\\ &= ma^2 - ah(m) + a^{-1}h(m), \quad \text{(Lemmas 11 and 12)},\\ &= ma^2 - h(m),\end{aligned}$$
$$\Rightarrow ma^2 - \tfrac{1}{2}a < ma^2 - \tfrac{1}{2} < g(d(m)) < ma^2 - \tfrac{1}{2}a^{-1} < ma^2 - \tfrac{1}{2}a + 1,$$
$$\Rightarrow g(d(m)) = [ma^2 - \tfrac{1}{2}a] + 1 = S(m,1) + 1.$$

We can now combine the results of Lemmas 11, 12 and 13 to give the result:

Theorem 4. For $h(m) \in (-\tfrac{1}{2}, -\tfrac{1}{2}a^{-2})$,

$$d(m) = g(m) - 1, \quad S(m,1) = g(d(m)) - 1;$$

for $h(m) \in (-\tfrac{1}{2}a^{-2}, \tfrac{1}{2}a^{-1})$,

$$d(m) = g(m) - 1, \quad S(m,1) = g(d(m)) + 1;$$

and for $h(m) \in (\tfrac{1}{2}a^{-1}, \tfrac{1}{2})$,

$$d(m) = g(m) \qquad S(m,1) = g(d(m)) - 1.$$

We now turn to the problem of finding the values of i,j, so that $d(m) = S(i,j)$, for a given $m \in N$.

Lemma 14. If $d(m) = S(r,1)$, then $r = [ma^{-1} + \tfrac{1}{2}]$.

Proof. By Lemma 12,

$$\begin{aligned} k(d(m)) = 1 &\Rightarrow h(m) \in (-\tfrac{1}{2}, \tfrac{1}{2}a^{-3}),\\ &\Rightarrow d(m) = g(m) - 1, \quad \text{(Theorem 4)},\\ &= [ma + \tfrac{1}{2}] - 1,\\ &= ma - \tfrac{1}{2} - \epsilon, \quad 0 < \epsilon < 1.\end{aligned}$$

Also $S(r,1) = [ra^2 - \frac{1}{2}a]$, so $d(m) = S(r,1)$,

$$\Rightarrow ra^2 - \tfrac{1}{2}a - 1 < ma - \tfrac{1}{2} - \epsilon < ra^2 - \tfrac{1}{2}a,$$
$$\Rightarrow r < ma^{-1} + \tfrac{1}{2}a^{-1} + \tfrac{1}{2}a^{-2} - \epsilon a^{-2} < r + a^{-2},$$
$$\Rightarrow r < r + \epsilon a^{-2} < ma^{-1} + \tfrac{1}{2} < r + (1+\epsilon)a^{-2} < r + 2a^{-2} < r+1,$$
$$\Rightarrow r = [ma^{-1} + \tfrac{1}{2}].$$

Lemma 15. If $d(m) = S(r,2)$, then $r = [ma^{-2} + \frac{1}{2}]$.

Proof.
$$d(m) = S(r,2) \Rightarrow k(d(m)) = 2,$$
$$\Rightarrow h(m) \in (\tfrac{1}{2}a^{-3}, \tfrac{1}{2}), \quad \text{(Lemma 12)}.$$

Let $r = [ma^{-2} + \frac{1}{2}] = ma^{-2} + \frac{1}{2} - \epsilon$, $0 < \epsilon < 1$. Now

$$\begin{aligned}\epsilon &\equiv ma^{-2} + \tfrac{1}{2} \pmod 1,\\ &\equiv -ma + \tfrac{1}{2} \pmod 1,\\ &\equiv \tfrac{1}{2} - h(m) \pmod 1.\end{aligned}$$

But $\frac{1}{2}a^{-3} < h(m) < \frac{1}{2}$, so $\epsilon = \frac{1}{2} - h(m)$, and $r = ma^{-2} + h(m)$,

$$\begin{aligned}S(r,1) &= [ra^2 - \tfrac{1}{2}a],\\ &= [m + h(m)a^2 - \tfrac{1}{2}a].\end{aligned}$$

For $h(m) \in (\frac{1}{2}a^{-3}, \frac{1}{2}a^{-1})$, $-\frac{1}{2} < h(m)a^2 - \frac{1}{2}a < 0$,

$$\begin{aligned}\Rightarrow S(r,1) &= m-1\\ \Rightarrow S(r,2) &= g(S(r,1)),\\ &= g(m-1),\\ &= [ma - a + \tfrac{1}{2}],\\ &= [g(m) + h(m) - a + \tfrac{1}{2}].\end{aligned}$$

Now $g(m) - 1 < g(m) + h(m) - a + \frac{1}{2} < g(m) - \frac{1}{2}a$,

$$\begin{aligned}\Rightarrow S(r,2) &= g(m) - 1,\\ &= d(m) \text{ by Theorem 4.}\end{aligned}$$

For $h(m) \in (\frac{1}{2}a^{-1}, \frac{1}{2})$,

$$\begin{aligned}S(r,2) &= g(S(r,1)),\\ &= g(m),\\ &= d(m) \text{ by Theorem 4.}\end{aligned}$$

Lemma 16. $$S(m,1) = d([ma + \tfrac{1}{2}a^{-2}]), \quad \forall m \in N.$$

Proof. Let $n = [ma + \frac{1}{2}a^{-2}] = ma + \frac{1}{2}a^{-2} - \epsilon$, $0 < \epsilon < 1$,

$$\Rightarrow ma + \tfrac{1}{2}a^{-2} - 1 < n < ma + \tfrac{1}{2}a^{-2},$$
$$\Rightarrow m = m + \tfrac{1}{2}a^{-3} - a^{-1} + \tfrac{1}{2} < na^{-1} + \tfrac{1}{2} < m + \tfrac{1}{2}a^{-3} + \tfrac{1}{2} = m + a^{-1},$$
$$\Rightarrow m = [na^{-1} + \tfrac{1}{2}].$$

Also

$$\begin{aligned}\epsilon &\equiv ma + \tfrac{1}{2}a^{-2} \pmod 1,\\ &\equiv h(m) + \tfrac{1}{2}a^{-2} \pmod 1.\end{aligned}$$

Hence

$$\begin{aligned}\epsilon &= h(m) + \tfrac{1}{2}a^{-2} + 1 \quad \text{for} \quad h(m) \in (-\tfrac{1}{2}, -\tfrac{1}{2}a^{-2}),\\ \epsilon &= h(m) + \tfrac{1}{2}a^{-2} \quad \text{for} \quad h(m) \in (-\tfrac{1}{2}a^{-2}, \tfrac{1}{2}).\end{aligned}$$

Further,

$$\begin{aligned}h(n) &\equiv na \pmod 1,\\ &\equiv ma^2 + \tfrac{1}{2}a^{-1} - \epsilon a \pmod 1,\\ &\equiv h(m) + \tfrac{1}{2}a^{-1} - \epsilon a \pmod 1,\end{aligned}$$

For

$$h(m) \in (-\tfrac{1}{2}, -\tfrac{1}{2}a^{-2}), \quad \epsilon = h(m) + \tfrac{1}{2}a^{-2} + 1,$$

$$\begin{aligned}\Rightarrow h(n) &\equiv -a^{-1}h(m) - a \pmod 1,\\ \Rightarrow h(n) &= -a^{-1}h(m) - a + 1,\\ &= -a^{-1}(h(m)-1),\\ \Rightarrow h(n) &\in (-\tfrac12, -\tfrac12 a^{-1}).\end{aligned}$$

For $h(m) \in (-\tfrac12 a^{-2}, \tfrac12)$, $\epsilon = h(m) + \tfrac12 a^{-2}$,

$$\begin{aligned}\Rightarrow h(n) &\equiv -a^{-1}h(m) \pmod 1,\\ \Rightarrow h(n) &= -a^{-1}h(m),\\ \Rightarrow h(n) &\in (-\tfrac12 a^{-1}, \tfrac12 a^{-3}).\end{aligned}$$

Hence in either case $h(n) \in (-\tfrac12, \tfrac12 a^{-3})$, so applying Lemma 14,

$$S(m,1) = S([na^{-1} + \tfrac12], 1) = d(n) = d([ma + \tfrac12 a^{-2}]).$$

Lemma 17. $\quad S(m,2) = d([ma^2 - \tfrac12 a^{-1}]), \quad \forall m \in N.$

Proof. Let $n = [ma^2 - \tfrac12 a^{-1}] = ma^2 - \tfrac12 a^{-1} - \epsilon, \quad 0 < \epsilon < 1,$

$$\begin{aligned}&\Rightarrow ma^2 - \tfrac12 a^{-1} - 1 < n < ma^2 - \tfrac12 a^{-1},\\ &\Rightarrow m < na^2 + \tfrac12 a^{-3} + a^{-2} = na^{-2} + \tfrac12 < m + a^{-2},\\ &\Rightarrow m = [na^{-2} + \tfrac12].\end{aligned}$$

Also

$$\begin{aligned}\epsilon &\equiv ma^2 - \tfrac12 a^{-1} \pmod 1,\\ &\equiv h(m) - \tfrac12 a^{-1} \pmod 1.\end{aligned}$$

Hence

$$\begin{aligned}\epsilon &= h(m) - \tfrac12 a^{-1} + 1 \quad \text{for} \quad h(m) \in (-\tfrac12, \tfrac12 a^{-1}),\\ \epsilon &= h(m) - \tfrac12 a^{-1} \quad \text{for} \quad h(m) \in (\tfrac12 a^{-1}, \tfrac12).\end{aligned}$$

Further

$$\begin{aligned}h(n) &\equiv na \pmod 1,\\ &\equiv ma^3 - \tfrac12 - \epsilon a \pmod 1,\\ &\equiv 2ma - \tfrac12 - \epsilon a \pmod 1,\\ &\equiv 2h(m) - \tfrac12 - \epsilon a \pmod 1.\end{aligned}$$

For $h(m) \in (-\tfrac12, \tfrac12 a^{-1})$, $\epsilon = h(m) - \tfrac12 a^{-1} + 1$,

$$\begin{aligned}\Rightarrow h(n) &\equiv a^{-2}h(m) - a \pmod 1,\\ \Rightarrow h(n) &= a^{-2}h(m) - a + 2,\\ &= a^{-2}(1 + h(m)),\\ \Rightarrow h(n) &\in (\tfrac12 a^{-2}, \tfrac12).\end{aligned}$$

For $h(m) \in (\tfrac12 a^{-1}, \tfrac12)$, $\epsilon = h(m) - \tfrac12 a^{-1}$,

$$\begin{aligned}\Rightarrow h(n) &\equiv h(m)(2-a) \pmod 1,\\ &= a^{-2}h(m) \pmod 1,\\ \Rightarrow h(n) &= a^{-2}h(m),\\ \Rightarrow h(n) &\in (\tfrac12 a^{-3}, \tfrac12 a^{-2}).\end{aligned}$$

Hence in either case $h(n) \in (\tfrac12 a^{-3}, \tfrac12)$, so applying Lemma 15, $S(m,2) = S([na^{-2} + \tfrac12], 2) = d(n) = d([ma^2 - \tfrac12 a^{-1}])$

These four Lemmas together with Lemma 12, give us Theorem 5.

Theorem 5.

$$\begin{aligned}d(m) &= S([ma^{-1} + \tfrac12], 1) \quad \text{if} \quad -\tfrac12 < h(m) < \tfrac12 a^{-3},\\ &= S([ma^{-2} + \tfrac12], 2) \quad \text{if} \quad -\tfrac12 a^{-3} < h(m) < \tfrac12,\\ S(m,1) &= d([ma + \tfrac12 a^{-2}]),\\ S(m,2) &= d([ma^2 - \tfrac12 a^{-1}]), \quad \forall m \in N.\end{aligned}$$

We can note now from Lemma 10 that as $d(m) < ma - \tfrac12 a^{-1} < m(a+1) - \tfrac12 a^{-1} - 1 < d(m+1)$, the sequence $d(m)$ is strictly monotonic increasing and hence by Theorem 5 we establish Butcher's conjecture.

Theorem 6.

$$\{S(m,1) : m \in N\} \cup \{S(m,2) : m \in N\} = \{d(m) : m \in N\}.$$

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