FIBONACCI AND LUCAS NUMBERS AND THE COMPLEXITY OF A GRAPH

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1. TERMINOLOGY

In this note we shall use the following notation and terminology:

$$\text{the Fibonacci numbers } F_n : F_1 = F_2 = 1,$$
$$F_{n+2} = F_{n+1} + F_n, \quad n \geqslant 1;$$
$$\text{the Lucas numbers } \quad L_n : L_1 = 1, \; L_2 = 3,$$
$$L_{n+2} = L_{n+1} + L_n, \quad n \geqslant 1;$$

α, β: zeros of the associated auxiliary polynomial;

a *composition* of a positive integer n is a vector $(a_1, a_2, \cdots, a_k)$ of which the components are positive integers which sum to n;

a *graph* G, is an ordered pair (V,E), where V is a set of vertices, and E is a binary relation on V; the ordered pairs in E are called the edges of the graph.

a *cycle* is a sequence of three or more edges that goes from a vertex back to itself;

a graph is *connected* if every pair of vertices is joined by a sequence of edges;

a *tree* is a connected graph which contains no cycles;

a *spanning tree* of a graph is a tree of the graph that contains all the vertices of the graph;

two spanning trees are *distinct* if there is at least one edge not common to them both;

the *complexity*, $k(G)$, of a graph is the number of distinct spanning trees of the graph.

For relevant examples see Hilton [2] and Rebman [4], and for details see Harary [1].

2. RESULTS

Hilton and Rebman have used combinatorial arguments to establish a relation between the complexity of a graph and the Fibonacci and Lucas numbers. Rebman showed that

$$K(W_n) = L_{2n} - 2, \tag{2.1}$$

where W_n, the n-wheel, is a graph with $n+1$ vertices obtained from a cycle on n points by joining each of these n points to a further point.

Hilton also established this result and

$$L_{2n} - 2 = \sum_{\gamma(n)} (-1)^{k-1} \frac{n}{k} F_{2a_1} \cdots F_{2a_k}, \tag{2.2}$$

in which $\gamma(n)$ indicates summation over all compositions $(a_1, \cdots, a_k)$ of n, the number of components being variable. It is proposed here to prove (2.1) by a number theoretic approach.

To do so we need the following preliminary results which will be proved in turn:

$$F_{2n} = F_{2n+2} - 2F_{2n} + F_{2n-2}, \tag{2.3}$$

$$1 - 2x^2 + x^4 = \exp\left(-2 \sum_{m=1}^{\infty} x^{2m}/m\right), \tag{2.4}$$

(2.5)
$$\sum_{n=0}^{\infty} F_{2n}x^{2n} = x^2 \exp\left(\sum_{m=1}^{\infty} L_{2m}x^{2m}/m\right),$$

(2.6)
$$1+\sum_{n=0}^{\infty} F_{2n}x^{2n} = (1-2x^2+x^4)\exp\left(\sum_{m=1}^{\infty} L_{2m}x^{2m}/m\right),$$

(2.7)
$$1+\sum_{n=1}^{\infty} F_{2n}x^{2n} = \exp\left(\sum_{m=1}^{\infty} (L_{2m}-2)x^{2m}/m\right),$$

wherein it is assumed that all power series are considered formally.

3. PROOFS

Proof of (2.3).

$$\begin{aligned} F_{2n} &= F_{2n}+F_{2n-1}-F_{2n-1} \\ &= F_{2n+1}-F_{2n}+F_{2n}-F_{2n-1} \\ &= F_{2n+1}-F_{2n}+F_{2n-2} \\ &= F_{2n+2}-2F_n+F_{2n-2}\ . \end{aligned}$$

Proof of (2.4).

$$\begin{aligned} 1-2x^2+x^4 &= (1-x^2)^2 \\ &= \exp\ln(1-x^2)^2 \\ &= \exp(-2\ln(1-x^2)^{-1}) \\ &= \exp\left(-2\sum_{m=1}^{\infty} x^{2m}/m\right). \end{aligned}$$

Proof of (2.5).

$$\begin{aligned} \sum_{n=0}^{\infty} F_{2n}x^{2n} &= x^2/(1-3x^2+x^4) \\ &= x^2/(1-\alpha^2x^2)(1-\beta^2x^2) \end{aligned}$$

$$\begin{aligned} \ln\left(\sum_{n=0}^{\infty} F_{2n}x^{2n-2}\right) &= -\ln(1-\alpha^2x^2)(1-\beta^2x^2) \\ &= -\ln(1-\alpha^2x^2)-\ln(1-\beta^2x^2) \\ &= \sum_{m=1}^{\infty}\frac{\alpha^{2m}x^{2m}}{m}+\sum_{m=1}^{\infty}\frac{\beta^{2m}x^{2m}}{m} \\ &= \sum_{m=1}^{\infty}(\alpha^{2m}+\beta^{2m})x^{2m}/m \\ &= \sum_{m=1}^{\infty} L_{2m}x^{2m}/m\ . \end{aligned}$$

Thus

$$\sum_{n=0}^{\infty} F_{2n}x^{2n-2} = \exp\left(\sum_{m=1}^{\infty} L_{2m}x^{2m}/m\right) \quad\text{and}\quad \sum_{n=0}^{\infty} F_{2n}x^{2n} = \exp\left(\sum_{m=1}^{\infty} L_{2m}x^{2m}/m\right).$$

Proof of (2.6).

$$\sum_{n=0}^{\infty} F_{2n}x^{2n-2} = \sum_{n=1}^{\infty} F_{2n}x^{2n-2}$$

$$= \sum_{n=0}^{\infty} F_{2n+2}x^{2n}$$

$$= \exp\left(\sum_{m=1}^{\infty} L_{2m}x^{2m}/m\right)$$

$$\sum_{n=0}^{\infty} F_{2n-2}x^{2n} = -1 + \sum_{n=0}^{\infty} F_{2n}x^{2n+2}$$

$$= -1 + x^2 \sum_{n=0}^{\infty} F_{2n}x^{2n}$$

$$= -1 + x^4 \exp\left(\sum_{m=1}^{\infty} L_{2m}x^{2m}/m\right).$$

Now

$$\sum_{n=0}^{\infty} F_{2n}x^{2n} = \sum_{n=0}^{\infty} (F_{2n+2} - 2F_{2n} + F_{2n-2})x^{2n}.$$

So

$$1 + \sum_{n=1}^{\infty} F_{2n}x^{2n} = (1 - 2x^2 + x^4)\exp\left(\sum_{m=1}^{\infty} L_{2m}x^{2m}/m\right).$$

Proof of (2.7).

$$1 + \sum_{n=1}^{\infty} F_{2n}x^{2n} = (1-x^2)^2 \exp\left(\sum_{m=1}^{\infty} L_{2m}x^{2m}/m\right) = \exp\left(\sum_{m=1}^{\infty} (L_{2m}-2)x^{2m}/m\right)$$

from (2.4).

4. MAIN RESULT

To prove the result (2.2) we let

$$W_n = \sum_{\gamma(n)} \frac{(-1)^{k-1}}{k} F_{2a_1} \cdots F_{2a_k}.$$

Then

$$\sum_{n=1}^{\infty} W_n x^{2n} = \sum_{n=1}^{\infty} \left\{\sum_{\gamma(n)} \frac{(-1)^{k-1}}{k} F_{2a_1} \cdots F_{2a_k}\right\} x^{2n}$$

$$= \sum_{k=1}^{\infty} -\left(-\sum_{n=1}^{\infty} F_{2n}x^{2n}\right)^k \Big/ k$$

$$= \ln\left(1 + \sum_{n=1}^{\infty} F_{2n}x^{2n}\right) = \sum_{n=1}^{\infty} (L_{2n}-2)x^{2n}/n$$

from which we get that

$$W_n = (L_{2n} - 2)/n$$

or

$$L_{2n} - 2 = \sum_{\gamma(n)} \frac{(-1)^{k-1}n}{k} F_{2a_1} \cdots F_{2a_k} .$$

These properties have been generalized elsewhere for arbitrary order recurrence relations [5].

Hoggatt and Lind [3] have also developed similar results in an earlier paper.

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EMBEDDING A GROUP IN THE p^{th} POWERS

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In a finite group G, the set of squares, cubes, or p^{th} powers in general, does not necessarily constitute a subgroup. However, we can always embed a finite group into the set of squares, cubes, or any p^{th} powers of another group.

A subgroup H of a group G is said to be a *subgroup of* p^{th} *powers* if for every $y \in H$, there is an $x \in G$ such that $x^p = y$.

Theorem. Every finite group G is isomorphic to a subgroup of p^{th} powers of some permutation group.

Proof. Let G be a finite group, and let P be an isomorphic permutation group on n elements, say $a_{11}, a_{12}, \cdots, a_{1n}$.

Consider a permutation group Q on pn elements

$$a_{11}, a_{12}, \cdots, a_{1n}; \quad a_{21}, a_{22}, \cdots, a_{2n}; \quad \cdots, \quad a_{p1}, a_{p2}, \cdots, a_{pn},$$

defined in the following manner: For any permutation

$$\sigma = (a_{1i_1} a_{1i_2} \cdots a_{1i_k}) \cdots (a_{1j_1} a_{1j_2} \cdots a_{1j_m})$$

in P corresponds the permutation

$$\begin{aligned}\hat{\sigma} = {} & (a_{1i_1} a_{1i_2} \cdots a_{1i_k})(a_{2i_1} a_{2i_2} \cdots a_{2i_n}) \cdots (a_{pi_1} a_{pi_2} \cdots a_{pi_k}) \\ & \cdots (a_{1j_1} a_{1j_2} \cdots a_{1j_m})(a_{2j_2} \cdots a_{2j_m}) \cdots (a_{pj_1} a_{pj_2} \cdots a_{pj_m})\end{aligned}$$

in the symmetric group S_{pn}. Q is clearly isomorphic to P and each elemenr in Q is the p^{th} power of an element in S_{pn}. In fact, $\hat{\sigma} = \tau^p$, where

$$\begin{aligned}\tau = {} & (a_{1i_1} a_{2i_1} \cdots a_{pi_1} a_{1i_2} a_{2i_2} \cdots a_{pi_2} \cdots a_{1i_k} a_{2i_k} \cdots a_{pi_k}) \\ & \cdots (a_{1j_1} a_{2j_1} \cdots a_{pj_1} a_{1j_2} a_{2j_2} \cdots a_{pj_2} \cdots a_{1j_m} a_{2j_m} \cdots a_{pj_m}).\end{aligned}$$

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