ON THE MULTIPLICATION OF RECURSIVE SEQUENCES

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1. INTRODUCTION

The object of this note is to generalize the results of Catlin [1] and Wyler [3] for the multiplication of recurrences. They studied second-order recurrences whereas the aim here is to set up definitions for their arbitrary order analogues.

The work is also related to that of Peterson and Hoggatt [2]. They considered a type of multiplication of series in their exposition of the characteristic numbers of Fibonacci-type sequences. In the last section of this paper we see how a definition of a characteristic arises from the earlier definition of multiplication.

We define an arbitrary order recursive sequence $\{W_n\}$ by the recurrence relation

$$W_n = \sum_{j=1}^{r} (-1)^{j+1} P_j W_{n-j}, \qquad n > r, \tag{1.1}$$

in which the P_j are arbitrary integers, and there are suitable initial values, $W_1, W_2, \cdots, W_r$. (Suppose $W_n = 0$ for $n \leqslant 0$.)

We shall need to consider some particular cases of these as well as some results associated with the product sums of the roots, a_t, of the associated auxiliary equation

$$a_t^r = \sum_{j=1}^{r} (-1)^{j+1} P_j a_t^{r-j} . \tag{1.2}$$

2. PRODUCT SUMS

We define the product sum

$$S_{tm} = \sum_{j \neq t} a_{j_1} a_{j_2} \cdots a_{j_m}$$

with $S_{to} = 1$. For example, when $r = 3$,

$$S_{31} = a_1 + a_2 \quad \text{and} \quad S_{32} = a_1 a_2 .$$

Some results we shall use now follow.

$$S_{tm} = P_m - a_t S_{t,m-1} . \tag{2.1}$$

Proof.

$$P_m - a_t S_{t,m-1} = \sum a_{j_1} a_{j_2} \cdots a_{j_m} - a_t \sum_{j \neq t} a_{j_1} a_{j_2} \cdots a_{j_m} = \sum_{j \neq t} a_{j_1} a_{j_2} \cdots a_{j_m} .$$

For example, when $r = 3$,

$$P_2 - a_1 S_{11} = a_1 a_2 + a_2 a_3 + a_3 a_1 - a_1 (a_2 + a_3) = a_2 a_3 = S_{12} .$$

$$S_{tr} = 0 \tag{2.2}$$

Proof.
$$P_j = S_{tj} + a_t S_{t,j-1}$$

$$\sum_{j=1}^{r} (-1)^{j+1} P_j a_t^{r-j} = \sum_{j=1}^{r} (-1)^{j+1} S_{tj} a_t^{r-j} - \sum_{j=1}^{r} (-1)^{j+1} S_{t,j-1} a_t^{r-j+1} \;;$$

that is

$$a_t^r = S_{tr} + S_{to} a_t^r \,,$$

which yields the result.

We note out of interest that:

$$S_{tm} = \sum_{j=0}^{m} (-1)^{m-j} P_j a_t^{m-j}, \qquad P_0 = 1 \tag{2.3}$$

Proof. We use induction on m.

$$S_{t0} = 1, \quad S_{t1} = P_1 - a_t, \quad \cdots,$$

$$S_{tm} = P_m - a_t S_{t,m-1} = P_m - a_t P_{m-1} + a_t^2 S_{t,m-2} = \sum_{j=0}^{m} (-1)^{m-j} P_j a_t^{m-j}.$$

$$\sum_{j=0}^{r-1} (-1)^j S_{tj} A_{n+r-j} = a_t^n \sum_{j=0}^{r-1} (-1)^j S_{tj} A_{r-j}, \qquad n \geqslant 0. \tag{2.4}$$

Proof. We use induction on n. When n is zero, the result is obvious. Suppose the result is true for $n = 1, 2, \cdots, k-1$. Then

$$\begin{aligned}
\sum_{j=0}^{r-1} (-1)^j S_{tj} A_{k+r-j} &= A_{k+r} + \sum_{j=1}^{r-1} (-1)^j S_{tj} A_{k+r-j} \\
&= \sum_{j=1}^{r} (-1)^{j+1} P_j A_{k+r-j} + \sum_{j=1}^{r-1} (-1)^j S_{tj} A_{k+r-j} \\
&= (-1)^{r+1} P_r A_k + \sum_{j=1}^{r-1} (-1)^j (S_{tj} - P_j) A_{k+r-j} \\
&= (-1)^{r+1} a_t S_{t,r-1} A_k + \sum_{j=1}^{r-1} (-1)^{j-1} a_t S_{t,j-1} A_{k+r-j} \\
&= (-1)^{r-1} a_t S_{t,r-1} A_k + \sum_{j=0}^{r-2} (-1)^j a_t S_{tj} A_{k+r-j-1} \\
&= a_t \sum_{j=0}^{r-1} (-1)^j S_{tj} A_{k+r-j-1} \\
&= a_t^{k-r} \sum_{j=0}^{r-1} (-1)^j S_{tj} A_{r-j} \quad \text{(by the inductive hypothesis)},
\end{aligned}$$

and so the result follows. In particular, it follows that

$$\sum_{j=0}^{r-1} (-1)^j S_{tj} A_{n+r-j} = a_t \sum_{j=0}^{r-1} (-1)^j S_{tj} A_{n+r-j-1}. \tag{2.5}$$

Result (2.4) is a generalization of Wyler's:

$$A_{n+1} - a_1 A_n = a_2^n (A_1 - a_1 A_0).$$

For ease of notation we shall write

$$\sum (t, A_n) = \sum_{j=0}^{r-1} (-1)^j S_{tj} A_{n+r-j}.$$

3. MATRIX RESULTS

We define matrices with rows i and columns j, $1 \le i,\ j \le r$:

(3.1) $$W^{(n)} = [W_{n+r-i+j}],$$

(3.2) $$M = [(-1)^{i+j} P_{j-i}], \quad \text{with} \quad P_n = \begin{cases} 0 \text{ for } n < 0 \\ 1 \text{ for } n = 0 \end{cases},$$

(3.3) $$S^{(t)} = [(-1)^{i+j} S_{t,j-i}], \quad \text{with} \quad S_{tn} = 0 \text{ for } n < 0,$$

(3.4) $$E = [\delta_{i,j-1}] \quad \text{(Kronecker delta)},$$

(3.5) $$Q = [q_{ij}], \quad \text{with} \quad q_{ij} = \begin{cases} (-1)^{j+1} P_j & \text{for } i = 1 \\ \delta_{i-1,j} & \text{for } i > 1 \end{cases}.$$

It follows from definitions (3.2), (3.3) and result (2.1) that

$$M = [(-1)^{i+j} P_{j-i}] = [(-1)^{i+j} S_{t,j-1}] - a_t [(-1)^{i+j} S_{t,j-i-1}] = S^{(t)} - a_t E S^{(t)} = (I - a_t E) S^{(t)}.$$

It can be readily proved by induction on n that

(3.6) $$W^{(n)} = Q^n W^{(0)}.$$

Furthermore,

$$S^{(t)} A^{(0)} = [\Sigma (t, A_{j-i})],$$

and so by using property (2.5), we find

$$S^{(t)} A (I - a_t E) = [S_{1j} \Sigma (t, A_{1-i})].$$

4. MULTIPLICATION

We can define a product $\{A_n\}\{B_n\}$ of two of these sequences to be the sequence $\{C_n\}$:

(4.1) $$C^{(0)} = A^{(0)} M B^{(0)}.$$

It follows from result (2.4) that

(4.2) $$C^{(m+n)} = Q^m C^{(0)} Q^{n^T} = A^{(m)} M B^{(n)}.$$

We can see how these generalize Catlin and Wyler. When $r = 2$:

$$W^{(0)} = \begin{bmatrix} W_2 & W_3 \\ W_1 & W_2 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & -P_1 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} P_1 & -P_2 \\ 1 & 0 \end{bmatrix}.$$

Result (4.2) becomes

$$\begin{bmatrix} C_{m+n+2} & C_{m+n+3} \\ C_{m+n+1} & C_{m+n+2} \end{bmatrix} = \begin{bmatrix} A_{m+2} & A_{m+3} \\ A_{m+1} & A_{m+2} \end{bmatrix} \begin{bmatrix} 1 & -P_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} B_{n+2} & B_{n+3} \\ B_{n+1} & B_{n+2} \end{bmatrix}$$
$$= \begin{bmatrix} A_{m+2} & A_{m+3} - P_1 A_{m+2} \\ A_{m+1} & A_{m+2} - P_1 A_{m+1} \end{bmatrix} \begin{bmatrix} B_{n+2} & B_{n+3} \\ B_{n+1} & B_{n+2} \end{bmatrix}.$$

from which we get, after equating corresponding matrix entries:

$$C_{m+n+2} = A_{m+2} B_{n+2} - P_2 A_{m+1} B_{n+1},$$
$$C_{m+n+1} = A_{m+1} B_{n+2} + A_{m+2} B_{n+1} - P_1 A_{m+1} B_{n+1},$$

in which we have used the recurrence relation

$$A_{m+3} = P_1A_{m+2} - P_2A_{m+1}\ .$$

These results agree with Catlin and Wyler.

For $r = 3$, we have

$$W^{(0)} = \begin{bmatrix} W_3 & W_4 & W_5 \\ W_2 & W_3 & W_4 \\ W_1 & W_2 & W_3 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & -P_1 & P_2 \\ 0 & 1 & -P_1 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} P_1 & -P_2 & P_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Result (4.2) now becomes

$$\begin{bmatrix} C_{m+n+3} & C_{m+n+4} & C_{m+n+5} \\ C_{m+n+2} & C_{m+n+3} & C_{m+n+4} \\ C_{m+n+1} & C_{m+n+2} & C_{m+n+3} \end{bmatrix}$$

$$= \begin{bmatrix} A_{m+3} & A_{m+4} - P_1A_{m+3} & A_{m+5} - P_1A_{m+4} + P_2A_{m+3} \\ A_{m+2} & A_{m+3} - P_1A_{m+2} & A_{m+4} - P_1A_{m+3} + P_2A_{m+2} \\ A_{m+1} & A_{m+2} - P_1A_{m+1} & A_{m+3} - P_2A_{m+2} + P_2A_{m+1} \end{bmatrix} \cdot \begin{bmatrix} B_{n+3} & B_{n+4} & B_{n+5} \\ B_{n+2} & B_{n+3} & B_{n+4} \\ B_{n+1} & B_{n+2} & B_{n+3} \end{bmatrix}$$

from which we obtain, for example,

$$C_{m+n+3} = A_{m+3}B_{n+3} + A_{m+4}B_{n+2} - P_1A_{m+3}B_{n+2} + P_3A_{m+2}B_{n+1}\ .$$

We further obtain

$$\sum_{j=0}^{r-1} (-1)^j s_{tj}C_{r-j} = \sum_{i=0}^{r-1} (-1)^i s_{ti}A_{r-i} \sum_{j=0}^{r-1} (-1)^j s_{tj}B_{r-j}\ . \tag{4.3}$$

Proof. We premultiply each side of definition (4.1) by $S^{(t)}$:

$$S^{(t)}C^{(0)} = S^{(t)}A^{(0)}MB^{(0)} = S^{(t)}A^0(I - a_tE)S^{(t)}B^{(0)} = S_{ij}\Sigma(t, A_{1-i})S^{(t)}B^{(0)},$$

or

$$\begin{bmatrix} \Sigma(t, C_0) & \Sigma(t, C_1) & \cdots & \Sigma(t, C_{r-1}) \\ \Sigma(t, C_{-1}) & \Sigma(t, C_{-2}) & \cdots & \Sigma(t, C_{r-2}) \\ & \cdots & \cdots & \\ \Sigma(t, C_{1-r}) & \Sigma(t, C_{-r}) & \cdots & \Sigma(t, C_0) \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma(t, A_0) & 0 & \cdots & 0 \\ \Sigma(t, A_{-1}) & 0 & \cdots & 0 \\ & & \cdots & \\ \Sigma(t, A_{1-r}) & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \Sigma(t, B_0) & \Sigma(t, B_1) & \cdots & \Sigma(t, B_{r-1}) \\ \Sigma(t, B_{-1}) & \Sigma(t, B_{-2}) & \cdots & \Sigma(t, B_{r-2}) \\ & & \cdots & \\ \Sigma(t, B_{1-r}) & \Sigma(t, B_{-r}) & \cdots & \Sigma(t, B_0) \end{bmatrix}$$

and so,

$$\Sigma(t, C_0) = \Sigma(t, A_0)\,\Sigma(t, B_0),$$

as required. When $r = 2$, $t = 1$, result (4.3) becomes

$$(C_2 - a_2C_1) = (A_2 - a_2A_1)(B_2 - a_2B_1)$$

as in Wyler and Catlin. When $r = 3$, $t = 1$:

$$(C_3 - (a_2 + a_3)C_2 + a_2a_3C_1) = (A_3 - (a_2 + a_3)A_2 + a_2a_3A_1)(B_3 - (a_2 + a_3)B_2 + a_2a_3B_1)\ .$$

Using property (2.4), we get

$$\sum_{i=0}^{r-1} (-1)^i s_{ti}A_{m+r-i} \sum_{j=0}^{r-1} (-1)^j s_{tj}B_{n+r-j} = a_t^{m+n} \sum_{i=0}^{r-1} (-1)^i s_{ti}A_{r-i} \sum_{j=0}^{r-1} (-1)^j s_{tj}B_{r-j}$$

$$= a_t^{m+n} \sum_{j=0}^{r-1} (-1)^j s_{tj}C_{r-j} = \sum_{j=0}^{r-1} (-1)^j s_{tj}C_{m+n+r-j}\ ,$$

as a generalization of Wyler's:

$$C_{m+n+2} - a_1C_{m+n+1} = a_2^{m+n}(A_2 - a_1A_1)(B_2 - a_1B_1) = (A_{m+2} - a_1A_{m+1})(B_{n+2} - a_1B_{n+1}).$$

5. NORMS AND DUALS

As in Catlin, we can define norms and duals. We define the norm or characteristic of $\{W_n\}$ as

(5.1) $$N\{W_n\} = \prod_{t=1}^{r} \sum_{j=0}^{r-1} (-1)^j S_{tj} W_{r-j}\ .$$

For example, for the "basic" sequences $\{U_{s,n}\}$ which satisfy the recurrence relation (1.1) but have initial conditions

$$U_{s,n} = S_{s,n}, \qquad n = 1, 2, \cdots, r,$$

we have

$$N\{U_{s,n}\} = \prod_{t=1}^{r} \sum_{j=0}^{r-1} (-1)^j S_{tj} U_{s,r-j} = \prod_{t=1}^{r} (-1)^{r-s} S_{t,r-s}\ ;$$

in particular, $N\{U_{r,n}\} = 1$. (The "basic" properties are seen in

$$W_n = \sum_{s=1}^{r} U_{s,n} W_s\ ,$$

for instance.)

(5.2) $$N\{A_n\}N\{B_n\} = N\{A_n\}\{B_n\}\ .$$

Proof.

$$N\{A_n\}N\{B_n\} = \prod_{t=1}^{r} \sum_{i=0}^{r-1} (-1)^i S_{ti} A_{r-i} \sum_{j=0}^{r-1} (-1)^j S_{tj} B_{r-j} = \prod_{t=1}^{r} \sum_{j=0}^{r-1} (-1)^j S_{tj} C_{r-j} = N\{C_n\} = N\{A_n\}\{B_n\}.$$

As

$$\Sigma(t, C_0) = \Sigma(t, A_0)\Sigma(t, B_0)$$

is related to $C^{(0)} = A^{(0)}MB^{(0)}$, so is

$$N\{C_n\} = N\{A_n\}N\{B_n\}$$

related to $|C^{(0)}| = |A^{(0)}|\,|B^{(0)}|$.

When $r = 2$, we have in fact that

$$N\{W_n\} = \begin{vmatrix} W_2 & W_3 \\ W_1 & W_2 \end{vmatrix} = W_2^2 - W_1W_3 = (W_2 - a_1W_1)(W_2 - a_2W_1).$$

Furthermore, from definition (5.1) we have that

$$P_r^n N\{W_n\} = P_r^n \prod_{t=1}^{r} \sum_{j=0}^{r-1} (-1)^j S_{tj} W_{r-j} = \prod_{t=1}^{r} a_t^n \sum_{j=0}^{r-1} (-1)^j S_{tj} W_{r-j} = \prod_{t=1}^{r} \sum_{j=0}^{r-1} (-1)^j S_{tj} W_{n+r-j}$$

as a generalization of Wyler's:

$$W_{n+2}^2 - W_{n+1}W_{n+3} = P_2^n N\{W_n\}.$$

We can compare this with

$$|W^{(n)}| = |Q^n|\,|W^{(0)}| \quad \text{in Eq. (3.6)}$$

$$= P_r^n |W^0|.$$

Similarly, we can form a dual as in Catlin. Given the recursive sequence $\{W_n\}$, we form its dual $\{W_n^*\}$ from the initial values

$$Wn,\ \ n = 1, 2, \cdots, r:$$

(5.3) $$\underset{\sim}{w}^* = \left(I - \sum_{k=1}^{r-1} (E^T)^k\right)\underset{\sim}{w}$$

where $$w = [W_1, W_2, \cdots, W_r]^T,$$

and E is the nilpotent matrix of order r defined in (3.4). For example, when $r = 2$,

$$\begin{bmatrix} W_1^* \\ W_2^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} W_1 \\ W_2 \end{bmatrix},$$

and $$W_1^* = W_1, \quad W_2^* = W_2 - W_1,$$

as in Catlin. When $r = 3$,

$$\begin{bmatrix} W_1^* \\ W_2^* \\ W_3^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix},$$

and so on. Essentially, what has been done here is to illustrate how the work for the second-order recurrences can be extended to any order. It may interest others to develop the algebra further by considering the canonical forms of elements in various extension fields and rings.

Another line of approach is to consider the treatment here as a generalization of Simson's (second-order) relation:

$$A_{n+1}^2 - A_nA_{n+2} = P_2^n N\{A_n\},$$

or, since $N\{F_n\} = 1$,

$$F_{n+1}^2 - F_nF_{n+2} = (-1)^n$$

for the Fibonacci numbers.

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