Solution by Paul S. Bruckman, Concord, California.

Since \( 2m < \log (k\sqrt{5}) / \log a < 2m + 1 \), it follows that \( a^{2m} < k\sqrt{5} < a^{2m+1} \); hence,

\[
a^{2m} - b^{2m} = a^{2m} - (-a^{-1})^{2m} < k\sqrt{5} < a^{2m+1} - (-a^{-1})^{2m+1} = a^{2m+1} - b^{2m+1},
\]

i.e.,

\[
F_{2m} < k < F_{2m+1}.
\]

Since \( \{F_n\}_1 \) is a non-decreasing sequence of positive integers, it follows that \( F_n < k \) for \( n = 1, 2, \ldots, 2m \), i.e., for \( 2m \) (distinct) values of \( n \).

Also solved by A. G. Shannon and the Proposer.

Also solved by the Proposer.

Therefore,

\[
d_{ij} = \sum_{k=1}^{n} c_k a_{kj} T = \sum_{k=1}^{n} \frac{(k-1)(j-1)}{k+1}.
\]

The effective limits of this summation are from \( k = 1 + \lceil \frac{s}{i} \rceil \) to \( \min (i,j) \). It will be convenient, however, to consider the upper limit to be equal to \( i \); if \( i > j \), the extra terms included vanish in any event. Therefore,

\[
d_{ij} = \sum_{k=1}^{i-1} \left( \frac{1}{i-1-k} \right)^{j-1} = \sum_{k=0}^{\lceil \frac{s}{i} \rceil} (i-1-k)(j-1-k).
\]

For convenience, let \( i-1 = r \), \( j-1 = s \).

Therefore,

\[
d_{ij} = \theta_{rs} = \sum_{k=0}^{\lceil \frac{s}{r} \rceil} \left( \frac{s-k}{r-k} \right).
\]

Let

\[
y = \sum_{r=0}^{\infty} \theta_{rs} x^r.
\]

Then

\[
y = \sum_{r=0}^{\infty} x^r \sum_{k=0}^{\lceil \frac{s}{r} \rceil} \frac{(r-k)}{k} \left( \frac{r}{r-k} \right) = \sum_{r=0}^{\infty} \sum_{k=2}^{\infty} x^r \frac{(r-k)}{k} \left( \frac{s}{r-k} \right) = \sum_{k=0}^{\infty} x^{2k} \sum_{r=0}^{\infty} x^r \frac{(s-k)}{r-k}.
\]

Thus,

\[
y = \sum_{k=0}^{\infty} \left( \frac{x}{k} \right)^{2k} \sum_{r=0}^{\infty} \left( \frac{s-k}{r} \right) x^r,
\]

by rearranging the combinatorial terms. Then,

\[
y = \sum_{k=0}^{\infty} \left( \frac{x}{k} \right)^{2k} (1+x)^{s-k} = (1+x)^s \sum_{k=0}^{\infty} \left( \frac{x^2}{1+x} \right)^k = (1+x)^s \left( 1 + \frac{x^2}{1+x} \right)^s,
\]

or:

\[
y = (1+x+x^2)^s.
\]

Therefore, \( d_{ij} \) is the coefficient of \( x^{i-1} \) in \( (1+x+x^2)^{i-1} \). From this, we may deduce that the \( d_{ij} \)'s satisfy the following recursion:

\[
d_{i+2,j+1} = d_{ij} + d_{i+1,j} + d_{i+2,j} \quad (i,j > 1); \quad d_{1,j} = 1, \quad d_{2,j} = j - 1 \quad (i > 1); \quad d_{i,1} = 0 \quad (i > 1).
\]

We may readily construct a matrix (of unspecified dimensions), whose \( j^{th} \) column is composed of the coefficients of \( (1+x+x^2)^{j-1} \), written in correspondence to the ascending powers of \( x \), beginning with \( x^0 \). For any given \( i \), \( d_{ii} = 0 \) for all \( i > 2j \) (since \( (1+x+x^2)^{j-1} \) contains \( 2j-1 \) non-zero terms).

Also solved by the Proposer.