ADVANCED PROBLEMS AND SOLUTIONS

Edited By
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-281 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Consider matrix equation

(a)
$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^n = \begin{pmatrix} A_n & B_n & C_n \\ D_n & E_n & G_n \\ H_n & I_n & J_n \end{pmatrix} \quad (n \geqslant 1).$$

Identify $A_n, B_n, C_n, \cdots, J_n$.

Consider matrix equation

(b)
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} A_n' & B_n' & C_n' \\ D_n' & E_n' & G_n' \\ H_n' & I_n' & J_n' \end{pmatrix} \quad (n \geqslant 1).$$

Identify $A_n', B_n', C_n', \cdots, J_n'$.

H-282 Proposed by H. W. Gould and W. E. Greig, West Virginia University.

Prove

$$\sum_{n=1}^{\infty} \frac{a^{2n}}{a^{4n}-1} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{a^{2k}-1},$$

where $a = (1+\sqrt{5})/2$, and determine which series converges the faster.

H-283 Proposed by D. Beverage, San Diego Evening College, San Diego, California.

Define $f(n)$ as follows:

$$f(n) = \sum_{k=0}^{n} \binom{n+k}{n} \left(\frac{1}{2}\right)^{n+k} \quad (n \geqslant 0).$$

Express $f(n)$ in closed form.

H-284 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

(A generalization of R. G. Buschman's H-18)

Show that

(a)
$$\sum_{k=0}^{n} \binom{n}{k} F_{rk} L_{rn-rk} = 2^n F_{rn} \quad \text{or} \quad (F^r + L^r)^n \doteq (2F^r)^n$$
(Umbral notation)

(b) $$\sum_{k=0}^{n} \binom{n}{k} L_{rk} L_{rn-rk} = 2^n L_{rn} + 2L_r^n \quad \text{or} \quad (L^r + L^r)^n \doteq (2L^r)^n + 2L_r^n$$

(c) $$\sum_{k=0}^{n} \binom{n}{k} F_{rk} F_{rn-rk} = \frac{(2^n L_{rn} - 2L_r^n)}{D}$$

Note. The generalization is valid for all Type I quadratic real fields, i.e, for $D = 5, 13, 29, 53, 61, \cdots$.

Remark on Problem H-123 by Henry Gould, West Virginia University.

The proposer's solution, *Fibonacci Quart.* 7 (1969), No. 2, 177–178, uses Stirling number expansions of factorials and powers. Since, however, it is true that

(1) $$\sum_{m=k}^{n} \mathcal{S}_n^{(m)} S_m^{(k)} = \delta_n^k = \begin{cases} 0, & k \neq n, \\ 1, & k = n, \end{cases}$$

then, for perfectly arbitrary F_k, and Fibonacci numbers in particular,

$$\sum_{m=0}^{n} \sum_{k=0}^{m} \mathcal{S}_n^{(m)} S_m^{(k)} F_k = \sum_{k=0}^{n} F_k \sum_{m=k}^{n} \mathcal{S}_n^{(m)} S_m^{(k)} = \sum_{k=0}^{n} F_k \delta_n^k = F_n$$

as desired. It is also true that

(2) $$\sum_{m=k}^{n} S_n^{(m)} \mathcal{S}_m^{(k)} = \delta_n^k ,$$

so by the same argument we have the dual formula to the original problem:

(3) $$\sum_{m=0}^{n} \sum_{k=0}^{m} S_n^{(m)} \mathcal{S}_m^{(k)} F_k = F_n ,$$

and, what is more interesting, this and the original formula hold for any sequence $\{F_n, n \geqslant 0\}$, the Fibonacci numbers really having nothing whatever to do with the truth of the formulas.

Relations (1) and (2) are the standard orthogonality relations for the two kinds of Stirling numbers, and are implied by the two expansions

(4) $$(x)_n = \sum_{k=0}^{n} S(n,k) x^k$$

and

(5) $$x^n = \sum_{k=0}^{n} \mathcal{S}(n,k)(x)_k ,$$

where

$$(x)_n = x(x-1)(x-2)\cdots 3\cdot 2\cdot 1, \quad \text{with } (x)_0 = 1.$$

Expansions (4)–(5) of course are the ones used by the proposer in his solution of his problem. Formulas (1) and (2) are both in Jordan's "Calculus of Finite Differences," page 184, the same source quoted by Lind for formulas (4)–(5). The essential point I am making is the generality of formulas (1)–(2) as opposed to the original solution.

EDITORIAL ACKNOWLEDGEMENT. Gregory Wulczyn, Bucknell University, submitted a solution for H-265 as well as an extensive partial solution for H-266.

SOLUTIONS

SUM SOLUTION

H-267 (Corrected) Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Show that

$$S(x) = \sum_{n=0}^{\infty} (kn+1)^{n-1} \frac{x^n}{n!},$$

where k is any integer and $0^0 \equiv 1$, satisfies

$$S(x) = e^{xS^k(x)}.$$

Solution by P. Bruckman, Concord, California.

We identify the given series as

(1) $$S(x) = \sum_{n=0}^{\infty} (kn+1)^{n-1} \frac{x^n}{n!}.$$

In "The H-Convolution Transform," V. E. Hoggatt, Jr., and Paul S. Bruckman, *Fibonacci Quarterly,* Vol. 13, No. 4, Dec. 1975, pp. 357–68, the following result is proved (where, to avoid confusion, we change the notation): Let

(2) $$f(x) = \sum_{i=0}^{\infty} a_{i:0}x^i,$$

(3) $$(f(x))^{j+1} = \sum_{i=0}^{\infty} a_{i:j}x^i,$$

where $f(0) \neq 0$, f is analytic about $x = 0$. Also, let

(4) $$G_{s,k}(x) \equiv G(x) = \sum_{i=0}^{\infty} \frac{s}{ki+s} a_{i:ki+s-1}x^i.$$

Then

(5) $$G(x) = f\{x(G(x))^k\}.$$

In particular, let

(6) $$f(x) = e^x, \quad s = 1.$$

Then

$$(f(x))^{j+1} = e^{(j+1)x} = \sum_{i=0}^{\infty} \frac{(j+1)^i x^i}{i!} = \sum_{i=0}^{\infty} a_{i:j}x^i,$$

which implies

(7) $$a_{i:j} = \frac{(j+1)^i}{i!}.$$

Hence,

$$\frac{s}{ki+s} a_{i:ki+s-1} = \frac{1}{ki+1} \frac{(ki+1)^i}{i!} = \frac{(ki+1)^{i-1}}{i!},$$

and also $G(x) = S(x)$, as given by (1). From (5), it now follows that

(8) $$\exp(xS^k(x)) = S(x).$$

Also solved by V. E. Hoggatt, Jr.

USE YOUR UMBRAL-AH

H-268 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$S_n(x) = \sum_{k=0}^{n} S(n,k)x^k,$$

where $S(n,k)$ denotes the Stirling number of the second kind defined by

$$x^n = \sum_{k=0}^{n} S(n,k)x(x-1)\cdots(x-k+1).$$

Show that

$$\begin{cases} xS_n(x) = \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j}S_{j+1}(x) \\ S_{n+1}(x) = x\sum_{j=0}^{n}\binom{n}{j}S_j(x) \end{cases}.$$

More generally evaluate the coefficients $C(n,k,j)$ in the expansion

$$x^kS_n(x) = \sum_{j=0}^{n+k} C(n,k,j)S_j(x) \qquad (k,n \geqslant 0).$$

Solution by P. Bruckman, Concord, California.

For the sake of typographical convenience, we make a slight change in notation. Let $S_1(n,k)$ and $S_2(n,k)$ denote the Stirling numbers of the first and second kinds, respectively, given by:

(1) $$x^{(n)} \equiv x(x-1)(x-2)\cdots(x-n+1) = \sum_{k=0}^{n} S_1(n,k)x^k,$$

(2) $$x^n = \sum_{k=0}^{n} S_2(n,k)x^{(k)}.$$

Also, we define $x^{(0)} \equiv 1$. The following orthogonality relation is satisfied by the Stirling numbers:

(3) $$\delta_{m:n} = \sum_{j=m}^{n} S_1(n,j)S_2(j,m).$$

Using (1)–(3), we may derive an explicit expression for the $c(n,k,j)$'s as follows:

$$x^kS_n(x) = \sum_{r=0}^{n} S_2(n,r)x^{r+k} = \sum_{r=0}^{n} S_2(n,r)\sum_{m=0}^{r+k} x^m\delta_{r+k:m} = \sum_{r=0}^{n} S_2(n,r)\sum_{m=0}^{r+k} x^m \sum_{j=m}^{r+k} S_1(r+k,j)S_2(j,m)$$

$$= \sum_{r=0}^{n} S_2(n,r)\sum_{j=0}^{r+k} S_1(r+k,j)\sum_{m=0}^{j} S_2(j,m)x^m = \sum_{r=0}^{n} S_2(n,r)\sum_{j=0}^{r+k} S_1(r+k,j)S_j(x)$$

$$= \sum_{j=0}^{n+k} S_j(x)\sum_{r=M}^{n} S_2(n,r)S_1(r+k,j),$$

where $M = \max(j-k, 0)$. Hence,

$$(4)\qquad c(n,k,j) = \sum_{r=M}^{n} S_2(n,r)S_1(r+k,j).$$

A more elegant algorithm for computing $c(n,k,j)$ may be derived by employing the umbral calculus, whereby $S_j(x)$ is replaced by S^j, and S is treated as an algebraic quantity. Returning to one of the relations preceding (4), and replacing true equality by "umbral equality," denoted by the symbol "$\triangleq$," we then have:

$$x^k S_n(x) = \sum_{r=0}^{n} S_2(n,r) \sum_{j=0}^{r+k} S_1(r+k,j)S_j(x) \triangleq \sum_{r=0}^{n} S_2(n,r) \sum_{j=0}^{r+k} S_1(r+k,j)S^j = \sum_{r=0}^{n} S_2(n,r)S^{(r+k)}$$

$$= S^{(k)} \sum_{r=0}^{n} S_2(n,r)(S-k)^{(r)} = S^{(k)}(S-k)^n.$$

More precisely, we have the generating function:

$$(5)\qquad \sum_{j=0}^{n+k} c(n,k,j)u^j = u^{(k)}(u-k)^n.$$

An alternative expression, derived by expanding $u^{(k)}$ in terms of $S_1(k,j)$'s, is the following:

$$(6)\qquad c(n,k,j) = \sum_{r=M}^{N} \binom{n}{r}(-k)^r S_1(k,j-r),$$

where M has been previously defined and $N = \min(j,n)$. Using the fact

$$(7)\qquad S_1(1,n) = \delta_{n:1},$$

we find in particular, from (6):

$$c(n,1,j) = \sum_{r=0}^{n} \binom{n}{r}(-1)^r S_1(1,j-r),$$

where the summation possibly includes undefined terms, which we define to be vanishing terms. Thus,

$$c(n,1,0) = \binom{n}{0}(-1)^0 S_1(1,0) = S_1(1,0) = 0; \quad c(n,1,n+1) = \binom{n}{n}(-1)^n S_1(1,1) = (-1)^n S_1(1,1) = (-1)^n;$$

if $1 \leqslant j \leqslant n$,

$$c(n,1,j) = \sum_{r=j-1}^{j} \binom{n}{r}(-1)^r S_1(1,j-r) = \binom{n}{j-1}(-1)^{j-1}S_1(1,1) + \binom{n}{j}(-1)^j S_1(1,0) = (-1)^{j-1}\binom{n}{j-1}.$$

Therefore, in all cases (i.e., for $j = 0, 1, \cdots, n+1$),

$$(8)\qquad c(n,1,j) = (-1)^{j-1}\binom{n}{j-1},$$

where the binomial coefficients $\binom{r}{s}$ are defined to vanish when $s < 0$ or $s > r$. Hence,

$$(9)\qquad xS_n(x) = \sum_{j=1}^{n+1} (-1)^{j-1}\binom{n}{j-1} S_j(x) = \sum_{j=0}^{n} \binom{n}{j}(-1)^j S_{j+1}(x).$$

By the well known technique of binomial inversion,

$$(10)\qquad S_{n+1}(x) = x \sum_{j=0}^{n} \binom{n}{j} S_j(x).$$

Also solved by F. Howard and the Proposer.

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