MORE FIBONACCI FUNCTIONS

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Recently there have appeared in this Quarterly a number of generalizations of the Fibonacci number F_n to functions $F(x)$, defined for all real x, and, in general, continuous everywhere.

For such a generalization two properties are particularly desirable:

(A) $$F(x) = F_n \quad \text{for} \quad x = n \text{ a natural number}$$

and

(B) $$F(x+2) = F(x) + F(x+1).$$

Spickerman [6] proved some general properties of functions satisfying (B).

Of the various generalizations Halsey's [1] does not generally satisfy (B) (see [7]) and even if defined for all real x, is not continuous at $x = 1$.

Heimer's function [2] satisfies (A) and (B) but is quasilinear. Elmore's function [3] is not a generalization in the above sense, it is a function of a natural number variable and a real variable.

Parker's [4] and Scott's [5] functions which are identical are "smooth curves," satisfy both (A) and (B) but can be generalized further.

Both take

$$F(x) = \text{Re}\left(\frac{\lambda^x - (-1)^x\lambda^{-x}}{\sqrt{(5)}}\right) = \frac{\lambda^x - \lambda^x \cos \pi x}{\sqrt{(5)}} ,$$

where

$$\lambda = \frac{1+\sqrt{5}}{2} .$$

It seems, however, that a lot is lost in taking only the real part of

$$\frac{\lambda^x - (-1)^x\lambda^{-x}}{\sqrt{(5)}} .$$

Clearly this complex function itself (we will call it F_x) satisfies (A), and also (B) for any complex number x. Also as the real part of F_x satisfies (B) so does the imaginary part and any linear combination of these.

If we let

$$F_1(x) = \text{Re}\,(F_x), \quad F_2(x) = I(F_x) = \frac{-\lambda^{-x} \sin \pi x}{\sqrt{(5)}} ,$$

for x real, then $F_1(x) + aF_2(x)$ satisfies (A) and (B) for each real number a.

Scott gives a number of identities concerning $F_1(x)$ and also concerning the corresponding Lucas function which we will call

$$L_1(x) = \text{Re}\,(L_x) = \text{Re}\,(\lambda^x + (-1)^x\lambda^{-x}) = \lambda^x + \lambda^{-x} \cos \pi x .$$

Of course $I(Lx) = -F_2(x)\sqrt{5}$.

We now list some easily derivable properties of $F_2(x)$ some of which relate it to $F_1(x)$:

$$F_2(x) \cdot F_2(-x) = \frac{-\sin^2 \pi x}{5}, \quad F_2(x+1) \cdot F_2(x-1) = F_2^2(x),$$

$$F_2(x + ¼) \cdot F_2(x - ¼) = F_2(2x)\,\frac{\cot 2\pi x}{2\sqrt{(5)}}, \quad F(x + ½) \cdot F_2(x - ½) = -F_2^2(x) \cot^2 \pi x,$$

$$F_1(x) = \frac{-\sin \pi x}{5F_2(x)} + F_2(x)\cot \pi x, \quad F_2(nx) = \frac{\sin n\pi x}{\sin^n \pi x}\,\frac{5^{(n/2)-1}F_2^n(x)}{(-1)^{n+1}}.$$

Another possible generalization of F_n for $x = n$ is $|F_x|$, which we will call $G_1(x)$. Thus

$$G_1(x) = |F_x| = \sqrt{F_1^2(x) + F_2^2(x)} = \frac{1}{\sqrt{(5)}}\sqrt{\lambda^{2x} - 2\cos \pi x + \lambda^{-2x}}$$

Another such function is

$$G_2(x) = \sqrt{F_1^2(x) - F_2^2(x)} = \frac{1}{\sqrt{(5)}}\sqrt{\lambda^{2x} - 2\cos \pi x + \lambda^{-2x}\cos 2\pi x}\,.$$

Clearly

$$kG_1(x) + (1-k)G_2(x) = F_n$$

when $x = n$ for all real k.

The following are some properties of these functions:

$$G_1^2(x+1) - G_1^2(x) = G_1^2(x+½) - 2/5\sin \pi x + 4/5\cos \pi x$$

$$G_1^2(2x) = 5G_1^4(x) + 4\cos \pi x G_1^2(x)$$

$$G_2^2(x) = (1/5)(L_1(2x) - 2\cos \pi x)$$

$$G_1^2(x) - G_2^2(x) = 2F_2^2(x).$$

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