FIBONACCI SINE SEQUENCES

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INTRODUCTION

The purpose of this note is to find all real numbers x such that $\lim_{n\to\infty} \sin u_n \pi x$ exists, where u_n is any sequence of integers satisfying the recurrence $u_n = u_{n-1} + u_{n-2}$ (u_0, u_1 are integers, not both zero).

We will show that the sequence $\{\sin u_n \pi x\}$ converges only to zero and this happens precisely when x is in an appropriate homothet of the set of integers in the quadratic number field $Q(\sqrt{5})$.

MAIN RESULTS

We will use the identity $\sin \alpha - \sin \beta = 2 \cos \tfrac{1}{2}(\alpha+\beta) \sin \tfrac{1}{2}(\alpha - \beta)$ to show that if the limit

$$\lim_n \sin u_n \pi x = \rho$$

exists, then $\rho = 0$.

Let $\alpha = u_{n+1}\pi x$, $\beta = u_{n-2}\pi x$, so that $\tfrac{1}{2}(\alpha+\beta) = u_n \pi x$, and $\tfrac{1}{2}(\alpha-\beta) = u_{n-1}\pi x$. The identity gives

$$\sin u_{n+1}\pi x - \sin u_{n-2}\pi x = 2 \sin u_{n-1}\pi x \cos u_n \pi x .$$

Therefore, if $\lim_n \sin u_n \pi x = \rho \neq 0$, then

$$\cos u_n \pi x = \frac{\sin u_{n+1}\pi x - \sin u_{n-2}\pi x}{2 \sin u_{n-1}\pi x}$$

shows that $\lim_n \cos u_n \pi x = 0$. However,

$$\sin u_{n+1}\pi x = \sin (u_n + u_{n-1})\pi x = \sin u_n \pi x \cos u_{n-1}\pi x + \cos u_n \pi x \sin u_{n-1}\pi x$$

implies $\lim_n \sin u_n \pi x = 0$, a contradiction.

Theorem 1. $\lim_n \sin u_n \pi x = 0$ iff

$$\lim_n \sin \phi^n \frac{\pi x}{\sqrt{5}} (u_0 + u_1 \phi) = 0, \quad \text{where} \quad \phi = \frac{1+\sqrt{5}}{2} .$$

Proof. Using Binet's formula for u_n, we have

$$\begin{aligned} \sin u_n \pi x &= \sin \frac{\pi x}{\sqrt{5}} \left\{ \phi^{n-1}(u_0 + u_1\phi) - (1-\phi)^{n-1}[u_0 + u_1(1-\phi)] \right\} \\ &= \sin \frac{\pi x}{\sqrt{5}} \phi^{n-1}(u_0 + u_1\phi) \cos \frac{\pi x}{\sqrt{5}} (1-\phi)^{n-1}[u_0 + u_1(1-\phi)] \\ &\quad - \sin \frac{\pi x}{\sqrt{5}} (1-\phi)^{n-1}[u_0 + u_1(1-\phi)] \cos \frac{\pi x}{\sqrt{5}} \phi^{n-1}(u_0 + u_1\phi) . \end{aligned}$$

Since $(1-\phi)^n \to 0$ as $n \to \infty$, the cosine in the first term tends to one, while the sine in the second term tends to zero, for any x. The theorem follows. ‖

Theorem 1 makes it plain that we must find the set B of all real x for which $\lim_n \sin \phi^n \pi x = 0$.

Theorem 2. B is the set of all numbers of the form $a + b\phi$, where a,b are integers.

Proof. We first observe that B is an additive subgroup of the real numbers, for

$$\sin \phi^n \pi(x-y) = \sin \phi^n \pi x \cos \phi^n \pi y - \cos \phi^n \pi x \sin \phi^n \pi y$$

shows that $x-y$ is in B if both x and y are in B. Now taking $u_0 = -1,\ u_1 = 2$ in Theorem 1 and observing that $2\phi - 1 = \sqrt{5}$, it is apparent that 1 is in B and hence the definition of B shows that ϕ is also in B. It follows that B contains every number of the form $a + b\phi$.

To prove that every member of B has this form, we adapt an argument from Cassels [1, p. 136]. If $\lim_n \sin \phi^n \pi x = 0$, then $\phi^n x = p_n + r_n$, where p_n is an integer and $\lim_n r_n = 0$. Let $s_n = p_{n+2} - p_{n+1} - p_n$, so that s_n is an integer. Then

$$\begin{aligned} s_n &= (\phi^{n+2}x - r_{n+2}) - (\phi^{n+1}x - r_{n+1}) - (\phi^n x - r_n) \\ &= \phi^n x(\phi^2 - \phi - 1) - (r_{n+2} - r_{n+1} - r_n) = -(r_{n+2} - r_{n+1} - r_n). \end{aligned}$$

Since $\lim_n r_n = 0$, we see that $\lim_n s_n = 0$. Since s_n is an integer, we must have $s_n = 0$ for all $n \geqslant n_0 \geqslant 1$. Thus $r_{n+2} = r_{n+1} + r_n$ for $n \geqslant n_0$. Using Binet's formula, we have for $n \geqslant n_0$,

$$r_n = \frac{r_{n_0+1} - (1-\phi)r_{n_0}}{\sqrt{5}}\ \phi^n - \frac{r_{n_0+1} - \phi r_{n_0}}{\sqrt{5}}\ (1-\phi)^n .$$

Because $\phi^n \to \infty$ and $(1-\phi)^n \to 0$ as $n \to \infty$, the coefficient of ϕ^n must be zero; in other words, $r_{n_0+1} = (1-\phi)r_{n_0}$. Thus, for $n \geqslant n_0$,

$$\begin{aligned} r_n &= \frac{\phi r_{n_0} - r_{n_0+1}}{\sqrt{5}}(1-\phi)^n = \frac{\phi r_{n_0} - (1-\phi)r_{n_0}}{\sqrt{5}}(1-\phi)^n \\ &= \frac{r_{n_0}}{\sqrt{5}}\ (2\phi - 1)(1-\phi)^n = r_{n_0}(1-\phi)^n . \end{aligned}$$

In particular, choosing $n = n_0$, we find $r_{n_0} = r_{n_0}(1-\phi)^{n_0}$. This implies $r_{n_0} = 0$, and therefore $\phi^{n_0}x = p_{n_0}$, so that

$$x = p_{n_0}(1/\phi)^{n_0} .$$

Using the facts that $1/\phi = \phi - 1$ and $\phi^2 = \phi + 1$, we see that $x = a + b\phi$ for suitable integers a and b. ||

CONCLUDING REMARKS

Combining Theorems 1 and 2, $\lim_n \sin u_n \pi x$ exists iff x is a member of the homothet

$$\frac{\sqrt{5}}{u_0 + u_1\phi}\ B = \left\{ \frac{\sqrt{5}}{u_0 + u_1\phi}\ x : x \in B \right\} .$$

It is well known [3; p. 201] that B is the set of all integers in the quadratic number field $Q(\sqrt{5})$ and this suggests comparison with other sine sequences. In [2], it is shown that $\lim_n \sin 2^n \pi x$ exists iff $2^{n_0}x$ is an integer for some $n_0 \in Z$. Here we have shown that $\lim_n \sin \phi^n \pi x$ exists iff $\phi^{n_0}x$ is an integer for some $n_0 \in Z$.

In closing, we suggest it would be of interest to consider the same problem for the sine sequences $\sin u_n \pi x$ when the u_n satisfies a recurrence $u_n = su_{n-1} + tu_{n-2}$, where s and t are positive integers.

REFERENCES

1. J. W. S. Cassels, *An Introduction to Diophantine Approximation,* Cambridge University Press, 1957.
2. M. B. Gregory and J. M. Metzger, "Sequences of Sines," *Delta,* Vol. 5 (1975), pp. 84–93.
3. I. Niven and H. Zuckerman, *An Introduction to the Theory of Numbers,* Third Ed., J. Wiley & Sons, New York, 1972.

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