SOME REMARKS ON A COMBINATORIAL IDENTITY

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SECTION 1

Let k, p, q, v be positive integers, $q < p < k$, n a non-negative integer and $\{\lambda_0 = 1, \lambda_1, \lambda_2, \ldots\}$ a sequence of indeterminates. Let $s(k, j)$ be the (signed) Stirling number of the first kind defined by

$$\sum_{j=0}^{k} s(k, j)x^j = x(x - 1) \ldots (x - k + 1).$$

Put

$$L(v, p, q) = \sum r_1 r_2 \ldots r_v \lambda_{d_1} \lambda_{d_2} \ldots \lambda_{d_v}, \tag{1.1}$$

where the summation is over all sets of integers $r_1, r_2, \ldots, r_v$ such that

$$p = r_0 \geq r_1 \geq r_2 \geq \ldots \geq r_v = p - q, \tag{1.2}$$

and

$$d_j = r_{j-1} - r_j \qquad (j = 1, 2, \ldots, v). \tag{1.3}$$

A. Ran [2] proved that

$$\sum_{j=0}^{k} s(k, j)L(j + n, p, q) \equiv 0 \tag{1.4}$$

identically, that is, for arbitrary $\lambda_1, \lambda_2, \lambda_3, \ldots$.

Hanani [1] has recently given another proof of (1.4). Hanani's proof is elementary but makes use of a rather difficult lemma.

The purpose of the present note is first to give another proof of (1.4) that makes use of the familiar recurrence

$$s(k + 1, j) = s(k, j - 1) - k \cdot s(k, j) \tag{1.5}$$

and the recurrence (2.2) below satisfied by $L(v, p, q)$. We show also that a result like (1.4) can be obtained for the more general sum

$$L_t(v, p, q) = \sum (r_1 r_2 \ldots r_v)^t \lambda_{d_1} \lambda_{d_2} \ldots \lambda_{d_v} \tag{1.6}$$

where again the summation is over all $r_1, r_2, \ldots, r_v$ that satisfy (1.2) and (1.3).

We have been unable to find a simple generating function for $L(v, p, q)$. However, we do give an operational formula for the sum

$$F_v(y, z) = \sum_{p=0}^{\infty} \sum_{q=0}^{p} L(v, p, q) y^p z^q. \tag{1.7}$$

See (4.5) below.

SECTION 2

In view of (1.2) and (1.3) we can rewrite (1.1) in the following form:

$$L(v, p, q) = \sum_{d_1 + \cdots + d_v = q} (p - d_1)(p - d_1 - d_2) \ldots (p - d_1 - \ldots - d_v)\lambda_{d_1} \lambda_{d_2} \ldots \lambda_{d_v} \tag{2.1}$$

where the summation is over all nonnegative integers $d_1, d_2, \ldots d_v$ satisfying $d_1 + \cdots + d_v = q$. Thus

$$\begin{aligned} L(v+1, p, q) &= \sum_{d + d_1 + \cdots + d_v = q} (p - d)(p - d - d_1) \ldots (p - d - d_1 - \ldots - d_v)\lambda_d \lambda_{d_1} \ldots \lambda_{d_v} \\ &= \sum_{d=0}^{q} (p - d)\lambda \sum_{d_1 + \cdots + d_v = q - d} (p - d - d_1) \ldots (p - d - d_1 - \ldots - d_v)\lambda_{d_1} \ldots \lambda_{d_v} \\ &= \sum_{d=0}^{q} (p - d)\lambda_d L(v, p - d, q - d), \end{aligned}$$

so that

$$L(v+1, p, q) = \sum_{d=0}^{q} (p - d)\lambda_d L(v, p - d, q - d). \tag{2.2}$$

In the next place, by (1.5) and (2.2),

$$\begin{aligned} \sum_{j=0}^{k+1} s(k+1, j)L(j+n, p, q) &= \sum_{j} \left\{s(k, j-1) - k \cdot s(k, j)\right\} L(j+n, p, q) \\ &= -k \sum_{s=0}^{j} s(k, j)L(j+n, p, q) \\ &\quad + \sum_{j=0}^{k} s(k, j)L(j+n+1, p, q) \\ &= -k \sum_{s=0}^{j} s(k, j)L(j+n, p, q) \\ &\quad + \sum_{j=0}^{k} s(k, j) \sum_{d=0}^{q} (p - d)\lambda_d L(j+n, p - d, q - d) \end{aligned}$$

(continued)

$$= -k\sum_{s=0}^{j} s(k,\ j)L(j+n,\ p,\ q)$$
$$+\sum_{d=0}^{q}(p-d)\lambda\sum_{j=0}^{k} s(k,\ j)L(j+n,\ p-d,\ q-d).$$

Hence, if we put

$$R(k,\ n,\ p,\ q) = \sum_{j=0}^{k} s(k,\ j)L(j+n,\ p,\ q), \tag{2.3}$$

it is clear that we have proved that

$$R(k+1,\ n,\ p,\ q) = -kR(k,\ n,\ p,\ q) + \sum_{d=0}^{q}(p-d)\lambda_d R(k,\ n,\ p-d,\ q-d). \tag{2.4}$$

In particular, for $k = p$, (2.4) reduces to

$$R(p+1,\ n,\ p,\ q) = \sum_{d=1}^{q}(p-d)\lambda_d R(p,\ n,\ p-d,\ q-d). \tag{2.5}$$

Taking $q = 0$ in (2.1) we get

$$L(v,\ p,\ 0) = \sum_{d_1+\cdots+d_v=0}(p-d_1)\ \ldots\ (p-d_1-\ldots-d_v)\lambda_{d_1}\lambda_{d_2}\ \cdots\ \lambda_{d_v} = p^v$$

as is also clear from (1.1). Thus substitution in (2.3) gives

$$R(k,\ n,\ p,\ 0) = \sum_{j=0}^{k} s(k,\ j)p^{j+n} = p^n\cdot p(p-1)\ \ldots\ (p-k+1),$$

so that

$$R(k,\ n,\ p,\ 0) = 0 \qquad (k > p), \tag{2.6}$$

while

$$R(k,\ n,\ p,\ 0) = \frac{p^n\cdot p!}{(p-k)!} \qquad (k \le p). \tag{2.7}$$

Finally, by (2.6) and repeated application of (2.4) and (2.5), we have

$$R(k,\ n,\ p,\ q) = 0 \qquad (k > p \ge q \ge 0). \tag{2.8}$$

SECTION 3

The above proof of (1.4) suggests the following generalization. Let $t \ge 1$ and define generalized Stirling numbers of the first kind by means of

$$\sum_{s=0}^{k} s_t(k,\ j)x^j = x(x-1^t)(x-2^t)\ \ldots\ \left(x-(k-1)^t\right). \tag{3.1}$$

Put

$$L_t(v, p, q) = \sum (r_1 r_2 \ldots r_v)^t \lambda_{d_1} \lambda_{d_2} \ldots \lambda_{d_v}, \tag{3.2}$$

where the summation is over all $r_1, r_2, \ldots, r_v$ such that

$$p = r_0 \geq r_1 \geq r_2 \geq \ldots \geq r_v = p - q,$$

and

$$d_j = r_{j-1} - r_j \qquad (j = 1, 2, \ldots, v).$$

Then

$$\sum_{j=0}^{k} s_t(k, j) L_t(j+n, p, q) = 0, \tag{3.3}$$

where

$$n \geq 0, \ k > p \geq q > 0. \tag{3.4}$$

The proof is exactly like the proof of (2.8) and will be omitted.

SECTION 4

Put

$$F_v(y, z) = \sum_{q=0}^{\infty} \sum_{p=q}^{\infty} L(v, p, q) y^p z^q, \ F_0(y, z) = \frac{1}{1-y} \tag{4.1}$$

and

$$\Lambda(z) = \sum_{d=0}^{\infty} \lambda_d z^d. \tag{4.2}$$

By (2.1),

$$L(v, p, q) = \sum_{d_1 + \cdots + d_v = q} (p - d_1)(p - d_1 - d_2) \ldots (p - d_1 - \ldots - d_v) \lambda_{d_1} \lambda_{d_2} \ldots \lambda_{d_v},$$

so that

$$F_v(y, z) = \sum_{d_1, \cdots, d_v = 0}^{\infty} \sum_{p \geq d_1 + \cdots + d_v} (p - d_1)(p - d_1 - d_2) \ldots (p - d_1 - \ldots - d_v) \cdot \lambda_{d_1} \lambda_{d_2} \ldots \lambda_{d_v} y^p z^{d_1 + \cdots + d_v}$$

$$= \sum_{d_1, \ldots, d_v = 0}^{\infty} \sum_{p \geq d_2 + \cdots + d_v} p(p - d_2) \ldots (p - d_2 - \ldots - d_v) \cdot \lambda_{d_1} \lambda_{d_2} \ldots \lambda_{d_v} y^p (yz)^{d_1} z^{d_2 + \cdots + d_v}$$

$$= \Lambda(yz) \sum_{d_2,\ldots,d_v=0}^{\infty} \sum_{p \geq d_2+\cdots+d_v} p(p-d_2)\ldots(p-d_2-\ldots-d_v) \cdot \lambda_{d_2}\ldots\lambda_{d_v} y^p z^{d_2+\cdots+d_v}.$$

Since

$$\sum_{d_2,\ldots,d_v=0}^{\infty} \sum_{p \geq d_2+\cdots+d_v} p(p-d_2)\ldots(p-d_2-\ldots-d_v)\lambda_{d_2}\ldots\lambda_{d_v} y^p z^{d_2+\cdots+d_v}$$

$$= (yD_y) \sum_{d_2,\ldots,d_v=0}^{\infty} \sum_{p \geq d_2+\cdots+d_v} (p-d_2)\ldots(p-d_2-\ldots-d_v)\lambda_{d_2}\ldots\lambda_{d_v} y^p z^{d_2+\cdots+d_v},$$

where $D_y = \partial/\partial y$, it follows that

$$F_v(y, z) = \big(y\Lambda(yz)D_y\big)F_{v-1}(y, z). \tag{4.3}$$

Iteration of (4.3) gives

$$F_v(y, z) = \big(y\Lambda(yz)D_y\big)^{v-1}F_1(y, z) \qquad (v \geq 1).$$

Moreover, by (2.1) and (4.1),

$$F_1(y, z) = \sum_{d=0}^{\infty}\sum_{p=d}^{\infty}(p-d)\lambda_d y^p z^d = \sum_{d=0}^{\infty}\sum_{p=0}^{\infty} p\lambda_d y^p (yz)^d$$

$$= \frac{y}{(1-y)^2}\Lambda(yz) = \big(y\Lambda(yz)D_y\big)F_0(y, z).$$

Hence we get

$$F_v(y, z) = \big(y\Lambda(yz)D_y\big)^v F_0(y, z) \qquad (v \geq 0) \tag{4.4}$$

and more generally

$$F_{v+n}(y, z) = \big(y\Lambda(yz)D_y\big)^v F_n(y, z) \qquad (v \geq 0,\ n \geq 0). \tag{4.5}$$

By (2.3)

$$R(k, n, p, q) = \sum_{j=0}^{k} s(k, j)L(j+n, p, q).$$

Thus

$$G_{k,n}(y, z) \equiv \sum_{q=0}^{\infty}\sum_{p=q}^{\infty} R(k, n, p, q)y^p z^q = \sum_{j=0}^{k} s(k, j)F_{j+n}(y, z)$$

$$= \sum_{j=0}^{k} s(k, j)\big(y\Lambda(yz)D_y\big)^j \cdot F_n(y, z).$$

Hence if we put

$$z^{(k)} = z(z-1)\ldots(z-k+1) = \sum_{j=0}^{k} s(k, j)z^j,$$

we have

$$G_{k,n}(y, z) = \left(y\Lambda(yz)D_y\right)^k \cdot F(y, z), \tag{4.6}$$

where by (2.8),

$$G_{k,n}(y, z) = \sum_{q=0}^{\infty} \sum_{\substack{p=q \\ p \geq k}}^{\infty} R(k, n, p, q)y^p z^q. \tag{4.7}$$

We remark that in the special case

$$\lambda_n = 1 \qquad (n = 0, 1, 2, \ldots), \tag{4.8}$$

(1.1) reduces to

$$L(v, p, q) = \sum r_1 r_2 \ldots r_n, \tag{4.9}$$

where the summation is over all r_1, r_2, ..., r_n such that

$$p \geq r_1 \geq r_2 \geq \ldots \geq r_v = p - q.$$

It is proved in the following article, "Enumeration of Certain Weighted Sequences," that, when (4.8) holds, $L(v, p, q)$ satisfies

$$L(v, p, q) = \frac{p-q}{q!} \sum_{s=0}^{q} (-1)^s \binom{q}{s}(p-s)^{v+q-1} \qquad (v \geq 1;\ p \geq q \geq 0). \tag{4.10}$$

REFERENCES

[1] H. Hanani, "A Combinatorial Identity," *The Fibonacci Quarterly*, Vol. 14 (1976), pp. 49-51.

[2] A. Ran, "One-Parameter Groups of Formal Power Series," *Duke Mathematical Journal*, Vol. 38 (1971), pp. 441-459.

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