ENUMERATION OF CERTAIN WEIGHTED SEQUENCES

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SECTION 1

The following problem has occurred somewhat incidentally in the preceding paper [1]. A complicated solution is implicit in the results of the paper. In the present paper we give a simple direct solution.

Let $v \geq 1$ and $p \geq q \geq 0$. Let $L(v, p, q)$ denote the sum

$$\sum r_1 r_2 \ldots r_v, \tag{1.1}$$

where the summation is over all $r_1, r_2, \ldots, r_v$ satisfying

$$p \geq r_1 \geq r_2 \geq \ldots \geq r_v = p - q. \tag{1.2}$$

To get a recurrence for $L(v, p, q)$, we observe that, for $v > 1$,

$$L(v, p, q) = (p - q)\sum r_1 r_2 \ldots r_{v-1},$$

where now

$$p \geq r_1 \geq r_2 \geq \ldots \geq r_{v-1} \geq p - q.$$

Hence

$$L(v, p, q) = (p - q)\sum_{k=0}^{q}\sum r_1 r_2 \ldots r_{v-1},$$

where, in the inner sum

$$p \geq r_1 \geq r_2 \geq \ldots \geq r_{v-1} = p - k.$$

It follows that

$$L(v, p, q) = (p - q)\sum_{k=0}^{q} L(v - 1, p, k) \qquad (v > 1). \tag{1.3}$$

Replacing q by $q - 1$ in (1.3), we get

$$L(v, p, q - 1) = (p - q + 1)\sum_{k=0}^{q-1} L(v - 1, p, k).$$

Combining this with (1.3) we get the recurrence

$$(p - q + 1)L(v, p, q) - (p - q)L(v, p, q - 1) = (p - q)(p - q + 1)L(v - 1, p, q). \tag{1.4}$$

We shall now think of p as an indeterminate and define

$$M(v, p, q) = \frac{L(v, p, q)}{p - q}. \tag{1.5}$$

Then (1.4) yields

$$M(v, p, q) = M(v, p, q-1) + (p-q)M(v-1, p, q) \qquad (v > 1), \tag{1.6}$$

together with the initial conditions

$$\begin{cases} M(1, p, q) = 1 & (q = 0, 1, 2, \ldots) \\ M(v, p, 0) = p^{v-1} & (v = 1, 2, 3, \ldots). \end{cases} \tag{1.7}$$

Clearly $M(v, p, q)$ is uniquely determined by (1.6) and (1.7). The first few values are easily computed

$v \backslash q$	0	1	2	3
1	1	1	1	1
2	p	$2p - 1$	$3p - 3$	$4p - 6$
3	p^2	$3p^2 - 3p + 1$	$6p^2 - 12p + 7$	$10p^2 - 30p + 25$
4	p^3	$4p^3 - 6p^2 + 4p - 1$	$10p^3 - 30p^2 + 35p - 15$	$20p^3 - 90p^2 + 150p - 90$

We shall show that generally

$$M(v, p, q) = \frac{1}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(p-s)^{v+q-1}. \tag{1.8}$$

For $v = 1$, (1.8) reduces to

$$\begin{aligned} M(1, p, q) &= \frac{1}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(p-s)^q \\ &= \frac{1}{q!}\sum_{s=0}^{q}(-1)^{q-s}\binom{q}{s}s^q = 1 \qquad (q = 0, 1, 2, \ldots), \end{aligned}$$

by well-known results from finite differences. Also by (1.8),

$$M(v, p, 0) = p^{v-1} \qquad (v = 1, 2, 3, \ldots).$$

Thus (1.7) is verified.

Now assume that (1.8) holds for all v, q such that

$$v + q < m. \tag{1.9}$$

Then, for $v + q = m$, we have

$$M(v, p, q-1) + (p-q)M(v-1, p, q)$$

$$= \frac{1}{(q-1)!}\sum_{s=0}^{q-1}(-1)^s\binom{q-1}{s}(p-s)^{v+q-2} + \frac{p-q}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(p-s)^{v+q-2}$$

$$= \frac{1}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(q-s)(p-s)^{v+q-2} + \frac{p-q}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(p-s)^{v+q-2}$$

$$= \frac{1}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(p-s)^{v+q-2}\big((q-s)+(p-q)\big)$$

$$= \frac{1}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(p-s)^{v+q-1}$$

$$= M(v, p, q).$$

Hence (1.8) holds for $v + q = m$, thus completing the induction.
Finally, by (1.5) and (1.8), we have

$$L(v, p, q) = \frac{p-q}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(p-s)^{v+q-1} = \frac{p-q}{q!}\Delta_p^q(p-q)^{v+q-1} \tag{1.10}$$

where Δ_p^q denotes the finite difference operator defined by

$$\Delta_p f(p) = f(p+1) - f(p), \quad \Delta_p^q f(p) = \Delta_p \cdot \Delta_p^{q-1} f(p).$$

For $p \geq q \geq 0$, $v \geq 1$, (1.10) evaluates the sum (1.1).

SECTION 2

For $p = q$, (1.8) reduces to

$$M(v, q, q) = \frac{1}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(q-s)^{v+q-1} = \frac{1}{q!}\sum_{s=0}^{q}(-1)^{q-s}\binom{q}{s}^{v+q-1},$$

so that

$$M(v, q, q) = S(v+q-1, q) \qquad (v \geq 1), \tag{2.1}$$

a Stirling number of the second kind. Generally, it follows from (1.8) that

$$M(v, p, q) = \frac{1}{q!}\sum_{s=0}(-1)^s\binom{q}{s}\sum_{t=0}^{v+q-1}\binom{v+q-1}{t}(p-q)^{v+q-t-1}(q-s)^t,$$

which gives

$$M(v, p, q) = \sum_{t=q}^{v+q-1} \binom{v+q-1}{t} (p-q)^{v+q-t-1} S(t, q). \tag{2.2}$$

It follows from (1.8) that

$$\sum_{n=0}^{\infty} M(n-q+1, p, q) \frac{z^n}{n!} = \frac{1}{q!} \sum_{s=0}^{\infty} (-1)^s \binom{q}{s} e^{(p-s)z} = \frac{1}{q!} e^{(p-q)z} (e^z - 1)^q ,$$

so that

$$\sum_{n=0}^{\infty} \sum_{q=0}^{n} M(n-q+1, p+q, q) x^q \frac{z^n}{n!} = e^{pz} \exp\left\{x(e^z - 1)\right\}. \tag{2.3}$$

For additional properties of the sum

$$\sum_{k=0}^{q} (-1)^k \binom{q}{k} (p-k)^n ,$$

see [2, Ch. 1].

SECTION 3

The results of §1 can be generalized in the following way. Let $t \geq 1$ and put

$$L\ (v, p, q) = \sum (r_1 r_2 \ldots r_v)^t , \tag{3.1}$$

where the summation is over all r , r , ..., r_v satisfying

$$p \geq r_1 \geq r_2 \geq \ldots \geq r_v = p - q.$$

Then, in the first place

$$L_t(v, p, q) = (p-q)^t \sum_{k=0}^{q} L_t(v-1, p, k) \qquad (v > 1). \tag{3.2}$$

It follows from (3.2) that

$$\begin{aligned} &(p-q+1)^t L_t(v, p, q) - (p-q)^t L_t(v, p, q-1) \\ &\quad = (p-q)^t (p-q+1)^t L_t(v-1, p, q) \qquad (v > 1). \end{aligned} \tag{3.3}$$

Hence

$$M_t(v, p, q) = M_t(v, p, q-1) + (p-q) M_t(v-1, p, q) \qquad (v > 1), \tag{3.4}$$

where

$$M_t(v, p, q) = \frac{L_t(v, p, q)}{(p - q)^t}$$

and

$$\begin{aligned} M_t(1, p, q) &= 1 \qquad (q = 0, 1, 2, \ldots) \\ M_t(v, p, 0) &= p^{t(v-1)} \qquad (v = 1, 2, 3, \ldots). \end{aligned} \tag{3.5}$$

As in §1, we are again thinking of p as an indeterminate.

By means of (3.4) and (3.5) it is easy to show that

$$M_t(v+1, p, q) = \sum_{i_0 + i_1 + \cdots + i_q = v} p^{i_0 t} (p-1)^{i_1 t} \ldots (p-q)^{t_q t}. \tag{3.6}$$

It then follows that

$$\sum_{v=0}^{\infty} M_t(v+1, p, q) z^v = \sum_{j=0}^{q} \left(1 - (p-j)^t z\right)^{-1}. \tag{3.7}$$

Now put

$$\prod_{j=0}^{q} \left(1 - (p-j)^t z\right)^{-1} = \sum_{j=0}^{q} \frac{A_j^{(t)}}{1 - (p-j)^t z}$$

where the $A_j^{(t)}$ are independent of z. Then

$$A_j^{(t)} = \prod_{\substack{i=0 \\ i \neq j}}^{q} \left(1 - (p-i)^t (p-j)^{-t}\right)^{-1} = (p-j)^{qt} \prod_{\substack{i=0 \\ i \neq j}}^{q} \left((p-j)^t - (p-i)^t\right)^{-1}. \tag{3.8}$$

Finally, we have

$$M_t(v+1, p, q) = \sum_{j=0}^{q} A_j (p-j)^{tv},$$

with $A_j^{(t)}$ given by (3.8).

For $t = 1$, (3.8) reduces to

$$A_j^{(1)} = (p-j)^q \prod_{\substack{i=0 \\ i \neq j}}^{q} (i-j)^{-1} = \frac{(-1)^j (p-j)^q}{j!(q-j)!} = \frac{(-1)^j}{q!} \binom{q}{j} (p-j)^q.$$

Hence (3.9) becomes

$$M_1(v+1, p, q) = \frac{1}{q!} \sum_{j=0}^{q} (-1)^j \binom{q}{j} (p-j)^{q+v},$$

in agreement with (1.8).

REFERENCES

[1] L. Carlitz, "Some Remarks on a Combinatorial Identity," *The Fibonacci Quarterly*, Vol. 16, No. 3 (June 1978), pp. 243-248.
[2] N. Nielsen, *Traite elementaire des nombres de Bernoulli* (Paris: Gauthier-Villars, 1923).

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