ENUMERATION OF PERMUTATIONS BY SEQUENCES

LEONARD CARLITZ
Duke University, Durham, North Carolina 27706

SECTION 1

André [2] discussed the enumeration of permutations by number of sequences; his results are reproduced in Netto [5, pp. 105-112]. Let $P(n, s)$ denote the number of permutations of $Z_n = \ldots 1, 2, \ldots, n \ldots$ with s ascending or descending sequences. For example, the permutation 24315 has the ascending sequences 24, 15 and the descending sequence 431; the permutation 613254 has the ascending sequences 13, 25 and the descending sequences 61, 32, 54. The total number of sequences is five. Generally, a permutation of Z_n has at most $n - 1$ sequences; such a permutation is called an *up-down* or *down-up* permutation according as it begins with an ascending or a descending sequence. Clearly, in this case all the sequences are of length two.

It is convenient to put

$$P(0, s) = \delta_{0,s}, P(1, s) = \delta_{0,s}. \tag{1.1}$$

André proved that $P(n, s)$ satisfies the recurrence

$$P(n+1, s) = sP(n, s) + 2P(n, s-1) + (n-s+1)P(n, s-2) \quad (n \geq 2). \tag{1.2}$$

With the convention $P(1, s) = \delta_{0,s}$, (1.2) holds for $n \geq 1$.

$P(n, s)$:

$n \backslash s$	0	1	2	3	4	5
1	1					
2		2				
3		2	4			
4		2	12	10		
5		2	28	58	32	
6		2	60	236	300	122

Let $A(n)$ denote the number of up-down and $B(n)$ the number of down up permutations of Z_n. Then

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$$A(n) = B(n) = \tfrac{1}{2}P(n,\ n-1) \qquad (n \ge 2). \tag{1.3}$$

Moreover, André [1] showed that

$$\sum_{n=0}^{\infty} A(n)\frac{z^n}{n!} = \sec z + \tan z, \tag{1.4}$$

with $A(0) = A(1) = 1$. Thus, a generating function for $P(n,\ n-1)$ is known; also, (1.4) yields an explicit formula for $A(n)$ and, therefore, also for $P(n,\ n-1)$.

A generating function for $P(n,\ s)$ has apparently not been found. We shall show that

$$\sum_{n=0}^{\infty}(1 - x^2)^{-n/2}\ \frac{z^n}{n!}\sum_{s=0}^{n} P(n+1,\ s)x^{n-s} = \frac{1-x}{1+x}\left(\frac{\sqrt{1-x^2} + \sin z}{x - \cos z}\right)^2. \tag{1.5}$$

We have been unable to find an explicit formula for $P(n,\ s)$. However, it follows from (1.2) and (1.3) that

$$\begin{aligned} P(n,\ n-2) &= 2A(n+1) - 4A(n) && (n \ge 2),\\ P(n,\ n-3) &= A(n+2) - 4A(n+1) - (n-5)A(n) && (n \ge 3), \end{aligned}$$

and so on. Generally, we have

$$P(n,\ n-s) = \sum_{j=1}^{s} f_{sj}(n)A(n+s-j) \qquad (n \ge s > 0),$$

where the $f_{sj}(n)$ are polynomials in n, $f_{s1}(n) = 1$. However, the $f_{sj}(n)$ are not evaluated.

If we let $P(n,\ r,\ s)$ denote the number of permutations of Z_n with r ascending and s descending sequences, it is easy to show that

$$\begin{cases} P(n,\ r,\ r) = P(n,\ 2r)\\ P(n,\ r,\ r-1) = P(n,\ r-1,\ r) = \tfrac{1}{2}P(n,\ 2r-1). \end{cases}$$

Moreover, $P(n,\ r,\ s) = 0$ unless $r = s,\ s+1$, or $s-1$. Also, permutations can be classified further according as they begin or end with either an ascending or descending sequence. This suggests the four enumerants

$$P_{++}(n,\ r,\ s),\quad P_{+-}(n,\ r,\ s),\quad P_{-+}(n,\ r,\ s),\quad P_{--}(n,\ r,\ s);$$

for precise definitions, see §5 below.

It is also of some interest to adapt another point of view. We define $\bar{P}(n,\ r,\ s)$ as the number of permutations π of Z_n with r ascending and s descending sequences in which we count an additional ascending sequence if π begins with a descending sequence, also an additional descending sequence if π ends with an ascending sequence. For the relation of $\bar{P}(n,\ r,\ s)$ to the other enumerants and a generating function, see §§5 and 6.

SECTION 2

Put

$$P_n(x) = \sum_{s=0}^{n-1} P(n, s)x^s \qquad (n \geq 1) \tag{2.1}$$

and

$$G(x, z) = \sum_{n=0}^{\infty} P_{n+1}(x)\frac{z^n}{n!}. \tag{2.2}$$

By (1.2) and (2.1),

$$\begin{aligned}
P_{n+2}(x) &= \sum_{s=0}^{n+1} P(n+2, s)x^s \\
&= \sum_{s=0}^{n+1} \left\{sP(n+1, s) + 2P(n+1, s-1) + (n-s+2)P(n+1, s-2)\right\}x^s \\
&= xP'_{n+1}(x) + 2xP_{n+1}(x) + \sum_{s=0} (n-x)P(n+1, s)x \\
&= xP'_{n+1}(x) = 2xP_{n+1}(x) + nx^2P_{n+1}(x) - x^3P'_{n+1}(x).
\end{aligned}$$

Hence

$$P_{n+2}(x) = (nx^2 + 2x)P_{n+1}(x) - (x^3 - x)P'_{n+1}(x) \qquad (n \geq 0). \tag{2.3}$$

It now follows from (2.2) that

$$\begin{aligned}
\frac{\partial G(x, z)}{\partial z} &= \sum_{n=0}^{\infty} P_{n+2}(x)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left\{(nx^2 + 2x)P_{n+1}(x) - (x^3 - x)P'_{n+1}(x)\right\}\frac{z^n}{n!} \\
&= 2xG(x, z) + x^2z\frac{\partial G(x, z)}{\partial z} - (x^3 - x)\frac{\partial G(x, z)}{\partial x}.
\end{aligned}$$

Thus

$$(x^3 - x)\frac{\partial G(x, z)}{\partial x} - (x^2z - 1)\frac{\partial G(x, z)}{\partial z} = 2xG. \tag{2.4}$$

The system

$$\frac{dx}{x^3 - x} = \frac{dz}{-x^2z + 1} = \frac{dG}{2xG} \tag{2.5}$$

has the integrals

$$z\sqrt{x^2 - 1} + \arcsin\frac{1}{x}, \quad \frac{x+1}{x-1}G. \tag{2.6}$$

It follows that

$$\frac{x+1}{x-1}G(x,\ z) = \phi\left(z\sqrt{x^2-1} + \arcsin\frac{1}{x}\right), \tag{2.7}$$

for some $\phi(u)$.

It is convenient to replace x by x^{-1} and z by xz, so that (2.7) becomes

$$\frac{1+x}{1-x}G(x^{-1},\ xz) = \phi\left(z\sqrt{1-x^2} + \arcsin x\right). \tag{2.8}$$

For $z = 0$, (2.8) reduces to

$$\frac{1+x}{1-x}G(x^{-1},\ 0) = \phi(\arcsin x).$$

Since $G(x^{-1},\ 0) = 1$, it follows at once that

$$\phi(u) = \frac{1+\sin u}{1-\sin u}. \tag{2.9}$$

Hence (2.8) becomes, on replacing z by $z/\sqrt{1-x^2}$,

$$\frac{1+x}{1-x}G\left(x^{-1},\ \frac{xz}{\sqrt{1-x^2}}\right) = \frac{1+\sin(z+\arcsin x)}{1-\sin(z+\arcsin x)}.$$

It can be verified that the right member is equal to

$$\left(\frac{\sqrt{1-x^2}+\sin z}{x-\cos z}\right)^2.$$

Therefore, we have

$$H(x,\ z) = \frac{1-x}{1+x}\left(\frac{\sqrt{1-x^2}+\sin z}{x-\cos z}\right)^2 \tag{2.10}$$

where

$$H(x,\ z) = G\left(x^{-1},\ \frac{xz}{\sqrt{1-x^2}}\right) = \sum_{n=0}^{\infty}(1-x^2)^{-n/2}\ \frac{z^n}{n!}\sum_{s=0}^{n}P(n+1,\ s)x^{n-s}. \tag{2.11}$$

SECTION 3

For $x = 0$, (2.10) reduces to

$$\sum_{n=0}^{\infty}P(n+1,\ n)\frac{z^n}{n!} = \frac{(1+\sin z)^2}{\cos^2 z} = 2\sec^2 z + 2\sec z\tan z - 1. \tag{3.1}$$

By (1.4),

$$\sum_{n=0}^{\infty}A(n+1)\frac{z^n}{n!} = \sec z\tan z + \sec^2 z \tag{3.2}$$

while, by (1.3),

$$\sum_{n=0}^{\infty}P(n+1,\ n)\frac{z^n}{n!} = 1 + 2\sum_{n=1}^{\infty}A(n+1)\frac{z^n}{n!} = -1 + 2\sum_{n=0}^{\infty}A(n+1)\frac{z^n}{n!}.$$

Hence (3.1) and (3.2) are in agreement.

We may rewrite (2.10) in the form

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{s=0}^{n}P(n+1,\ s)x^{n-s} = \frac{1-x}{1+x}\left(\frac{\sqrt{1-x^2}+\sin(z\sqrt{1-x^2})}{x-\cos(z\sqrt{1-x^2})}\right)^2 \tag{3.3}$$

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It is clear from the definition that

$$\sum_{s=0}^{n}P(n+1,\ s) = (n+1)! \tag{3.4}$$

Hence, for $x = 1$, the left-hand side of (3.3) should reduce to

$$\sum_{n=0}^{\infty}(n+1)z^n = (1-z)^{-2}.$$

As for the right-hand side of (3.3), we have

$$\frac{1-x}{1+x}\left\{\frac{(1-x^2)^{\frac{1}{2}} + z(1-x^2)^{\frac{1}{2}} - \frac{1}{3!}z\ (1-x^2)^{\frac{3}{2}} + \cdots}{x - 1 + \frac{1}{2!}z^2(1-x^2) - \frac{1}{4!}z^4(1-x^2)^2 + \cdots}\right\}^2$$

$$= \left\{\frac{1 + z - \frac{1}{3!}z^3(1-x^2) + \cdots}{1 - \frac{1}{2!}z^2(1+x) + \cdots}\right\}^2,$$

which reduces to

$$\left(\frac{1+z}{1-z^2}\right)^2 = (1-z)^{-2}. \tag{3.5}$$

Note also that for $x = -1$, we get $(1+z)^2$. It therefore follows from (3.3) that

$$\sum_{s=0}^{n}(-1)^{n-s}P(n+1,\ s) = 0 \qquad (n > 2). \tag{3.6}$$

This is a known result [2], [5].

Combining (3.6) with (3.4) gives

$$\sum_{2s\le n}P(n+1,\ 2s) = \sum_{2s\le n}P(n+1,\ 2s+1) = \tfrac{1}{2}(n+1)! \tag{3.7}$$

If we take $s = n$ in (1.2) we get $P(n+1, n) = 2P(n,\ n-1) + P(n,\ n-2)$. Thus it follows from (1.3) that

$$P(n,\ n-2) = 2A(n+1) - 4A(n) \qquad (n \ge 2). \tag{3.8}$$

Taking $s = n - 1$, we get

$$P(n+1, n-1) = (n-1)P(n, n-1) + 2P(n, n-2) + 2P(n, n-3),$$

which gives

$$P(n, n-3) = A(n+2) - 4A(n+1) - (n-5)A(n) \qquad (n \geq 3). \tag{3.9}$$

Next, taking $s = n - 2$, we get

$$P(n, n-4) = A(n+3) - 6A(n+2) - (3n-16)A(n+1) + (6n-18)A(n) \tag{3.10}$$

$$(n \geq 4).$$

Thus it appears that

$$P(n, n-s) = \sum_{j=1}^{s} f_{sj}(n)A(n+s-j) \qquad (n \geq s > 0), \tag{3.11}$$

where the $f_{sj}(n)$ are polynomials in n, $f_{s1}(n) = 1$. Indeed, using (1.2), we find that

$$sf_{s+1,j}(n) = f_{s,j}(n+1) - (n-s+1)f_{s-1,j-2}(n) - 2f_{s,j-1}(n). \tag{3.12}$$

However, it is not evident how to evaluate the $f_{s,j}(n)$ from this recurrence.
Returning to (2.10), if we replace x by $\cos x$, we get

$$\sum_{n=0}^{\infty} \frac{(z/\sin x)^n}{n!} \sum_{s=0}^{n} P(n+1, s)\cos^{n-s} x = \frac{1-\cos x}{1+\cos x}\left(\frac{\sin x + \sin z}{\cos x - \cos z}\right)^2.$$

Hence

$$\cot \frac{1}{2}x \sum_{n=0}^{\infty} \frac{(z/\sin x)^n}{n!} \sum_{s=0}^{n} P(n+1, s)\cos^{n-s} x = \cot^2 \frac{1}{2}(x-z). \tag{3.13}$$

Since the right-hand side of (3.13) is symmetric in x, z, it follows that

$$\frac{1}{2}x \sum_{n=0}^{\infty} \frac{(z/\sin x)^n}{n!} \sum_{s=0}^{n} P(n+1, \quad)\cos^{n-s} x \tag{3.14}$$

$$= \cot \frac{1}{2}z \sum_{n=0}^{\infty} \frac{(x/\sin z)^n}{n!} \sum_{s=0}^{n} P(n+1, x)\cos^{n-s} z.$$

It would be interesting to know whether there is some combinatorial result equivalent to (3.14).

SECTION 5

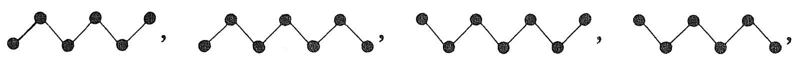
As a refinement of $P(n, s)$ we define $P(n, r, s)$ as the number of permutations of Z_n with r ascending and s descending sequences. It is evident that $P(n, r, s) = 0$ unless $r = s$, $s + 1$, or $s - 1$. Moreover, since a permutation can be read from left to right or right to left, we have

$$P(n, r, r-1) = P(n, r-1, r).$$

It accordingly follows that

$$\begin{cases} P(n,\ r,\ r) = P(n,\ 2r) \\ P(n,\ r,\ r - 1) = P(n,\ r - 1,\ r) = \frac{1}{2}P(n,\ 2r) \end{cases} \tag{5.1}$$

Now divide the permutations of Z into four nonoverlapping classes according as they begin or end with ascending or descending sequence. We denote the classes by C_{++}, C_{+-}, C_{-+}, C_{--}. The permutations in these classes have the appearance



(5.2)

respectively. Denote the corresponding enumerants by

$$P_{++}(n,\ r,\ s),\quad P_{+-}(n,\ r,\ s),\quad P_{-+}(n,\ r,\ s),\quad P_{--}(n,\ r,\ s).$$

Then we have the following equalities:

$$P_{++}(n,\ r,\ s) = P_{--}(n,\ s,\ r) \tag{5.3}$$

and

$$P_{+-}(n,\ r,\ s) = P_{-+}(n,\ s,\ r).$$

These relations follow on applying the transformation

$$b_i = n - a_i + 1 \qquad (i = 1,\ 2,\ \ldots,\ n)$$

to any permutation $(a_1,\ a_2,\ \ldots,\ a_n)$ of Z_n. Alternatively (5.3) follows on first reading a permutation of C_{++} from left to right and then from right to left.

In the next place, it is evident from (5.2) that $r = s + 1$ in C_{++}, $r = s$ in C_{++} or C_{--}, $r = s - 1$ in C_{--}. Thus

$$P_{+-}(n,\ r,\ s) = P_{-+}(n,\ r,\ s) = 0 \qquad (r \neq s), \tag{5.5}$$

$$P_{++}(n,\ r,\ s) = 0 \qquad (r \neq s + 1), \tag{5.6}$$

$$P_{--}(n,\ r,\ s) = 0 \qquad (r \neq s - 1). \tag{5.7}$$

Hence

$$\begin{cases} P_{+-}(n,\ r,\ r) = P_{-+}(n,\ r,\ r) = \frac{1}{2}P(n,\ 2r) \\ P_{++}(n,\ r,\ r - 1) = P_{--}(n,\ r - 1,\ r) = \frac{1}{2}P(n,\ 2r - 1). \end{cases} \tag{5.8}$$

In view of (5.8), generating functions for the four enumerants are implied by (2.10).

Another point of view is of some interest. Given a permutation $(a_1, a_2, \ldots, a_n)$ of Z_n, we adjoin virtual elements 0, 0′ : $(0, a_1, a_2, \ldots, a_n, 0')$. If $a_1 > a_2$, then $0a_1$ is counted as an additional ascending sequence; if however $a_1 < a_2$, the number of ascending sequences is unchanged. Similarly, if $a_{n-1} < a_n$, then $a_n 0'$ is counted as an additional descending sequence; if $a_{n-1} > a_n$, the number of descending sequences is unchanged. Also, let $P(n, r, s)$ denote the number of permutations of Z_n with r ascending and s descending sequences using these conventions. It follows at once that

$$\overline{P}(n, r, s) = 0 \qquad (r \neq s). \tag{5.9}$$

Moreover we have, by (5.8)

$$\begin{aligned}\overline{P}(n, r, r) = P_{+-}(n, r, r) &+ P_{-+}(n, r-1, r-1) \\ &+ P_{++}(n, r, r-1) + P_{--}(n, r-1, r).\end{aligned} \tag{5.10}$$

To illustrate (5.10), take $n = 4$, $r = 2$. The permutations are:

$$C_{++}\left\{\begin{matrix} 1 & 3 & 2 & 4 \\ 1 & 4 & 2 & 3 \\ 2 & 3 & 1 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 4 & 1 & 2 \end{matrix}\right. \qquad C_{--}\left\{\begin{matrix} 2 & 1 & 4 & 3 \\ 3 & 1 & 4 & 2 \\ 3 & 2 & 4 & 1 \\ 4 & 1 & 3 & 2 \\ 4 & 2 & 3 & 1 \end{matrix}\right.$$

For $n = 3$, $r = 2$, the permutations are:

$$C_{-+}\left\{\begin{matrix} 2 & 1 & 3 \\ 3 & 1 & 2 \end{matrix}\right.$$

For $n = 3$, $r = 1$:

$$C_{+-}\left\{\begin{matrix} 1 & 3 & 2 \\ 2 & 3 & 1 \end{matrix}\right. .$$

It follows form (5.8) and (5.10) that

$$\overline{P}(n, 2r) = P_{+-}(n, r, r) + P_{-+}(n, r-1, r-1) + P(n, 2r-1). \tag{5.11}$$

We have also

$$\overline{P}_n(x) = P_n^{+-}(x) + x^{-2}P_n^{-+}(x) + x^{-1}P_n^{++}(x) + x^{-1}P_n^{--}(x) \tag{5.12}$$

and

$$P_n(x) = P_n^{+-}(x) + P_n^{-+}(x) + P_n^{++}(x) + P_n^{--}(x), \tag{5.13}$$

where

$$P_n(x) = \sum_r P(n, r)x^{n-k}, \quad \overline{P}_n(x) = \sum_r \overline{P}(n, r, r)x^{n-2r},$$

$$P_n^{+-}(x) = \sum_r P_{+-}(n,\ r,\ r)x^{n-2r},$$

$$P_n^{++}(x) = \sum_r P_{++}(n,\ r,\ r-1)x^{n-2r-1},\ \text{etc.}$$

Note that $P_n(x)$ is not the same as the $P_n(x)$ of (2.1).

Comparison of (5.13) with (5.12) gives

$$\overline{P}_n(x) - x^{-1}P_n(x) = (1 - x^{-1})^2 P_n^{+-}(x). \tag{5.14}$$

SECTION 6

A generating function for $P(n,\ r,\ r)$ can be obtained rapidly by using a known result on the enumeration of permutations by maxima. Given the permutation $(a_1,\ a_2,\ \ldots,\ a_n)$ of Z_n, then a_k, $1 < k < n$, is a maximum if $a_{k-1} < a_k$, $a_k > a_{k-1}$. In addition, a_1 is a maximum if $a_1 > a_2$; a_n is a maximum if $a_{n-1} < a_n$. Let $M(n,\ m)$ denote the number of permutations of Z with m maxima.

Clearly if a permutation has m maxima in accordance with this definition, then it has exactly m ascending and m descending sequences and conversely. Thus

$$\overline{P}(n,\ r,\ r) = M(n,\ r). \tag{6.1}$$

A generating function for $M(n,\ k)$ is furnished by [3], [4]:

$$\sum_{n,k=0}^{\infty} M(n + 2k + 1,\ k + 1)\frac{u^n v^{2k}}{(n + 2k)!} \tag{6.2}$$

$$= \left\{\cosh\sqrt{u^2 - v^2} - \frac{u}{\sqrt{u^2 - v^2}}\sinh\sqrt{u^2 - v^2}\right\}^{-2}.$$

Making some changes in notation, this becomes

$$\sum_{n=0}^{\infty}(1 - x^2)^{-n/2}\frac{z^n}{n!}\sum_{2j \le n} M(n+1,\ j+1)x = \frac{1 - x^2}{(\sqrt{1 - x^2}\cos z - x\sin z)^2}. \tag{6.3}$$

Finally, in view of (6.1), we have

$$\sum_{n=0}^{\infty}(1 - x^2)^{-n/2}\frac{z^n}{n!}\sum_{2j \le n} P(n+1,\ j+1,\ j+1)x = \frac{1 - x^2}{(\sqrt{1 - x^2}\cos z - x\sin z)^2}. \tag{6.4}$$

If we put

$$H(x,\ z) = \sum_{n=0}^{\infty}(1 - x^2)^{-n/2}\frac{z^n}{n!}P_{n+1}(x),\quad H(x,\ z) = \sum_{n=0}^{\infty}(1 - x^2)^{-n/2}\frac{z^n}{n!}\overline{P}_{n+1}(x),$$

$$H^{+-}(x,\ z) = \sum_{n=0}^{\infty}(1 - x^2)^{-n/2}\frac{z^n}{n!}P_{n+1}^{+-}(x),$$

it follows from (5.14) that

$$x\overline{H}(x,\ z) - x^{-1}H(x,\ z) = (1 - x^{-1})^2 H^{+-}(x,\ z). \tag{6.5}$$

Therefore, by (2.10) and (5.14), we get

$$x^{-1}(1-x^2)H^{+-}(x,\ z) = \frac{x^2(1+x)^2}{\left(\sqrt{1-x^2}\cos z - x\sin z\right)^2} - \left(\frac{\sqrt{1-x^2} + \sin z}{x - \cos z}\right)^2. \tag{6.6}$$

Values of $P(n,\ r,\ s)$ for $n = 2,\ 3,\ 4$ follow.

$n = 2$:

$r \backslash s$	0	1
0	•	1
1	1	•

$n = 3$:

$r \backslash s$	0	1
0	•	1
1	1	4

$n = 4$:

$r \backslash s$	0	1	2
0	•	1	•
1	1	12	5
2	•	5	•

$n = 5$:

$r \backslash s$	0	1	2
0	•	1	•
1	1	28	29
2	•	29	32

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