KNIGHT'S TOUR REVISITED

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ABSTRACT

This paper shows that on any big enough rectangular chessboard there is a knight's path. If the number of squares is even there is a circuit.

A board is big enough if its smaller dimension is at least 5.

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Almost two hundred years ago Euler investigated the problem of whether it was possible to move a knight through every square of a chessboard once and return to the starting square. Euler demonstrated that this knight's tour was possible by displaying a chessboard with the required sequence of moves. He also generalized the problem by showing that there were other size boards on which the knight's tour was possible.

We must recall at this point that the standard chessboard has 64 squares arranged in 8 equal rows and columns. The knight is the chess piece that often looks like a horse. If the knight is on square (i, j), it is allowed to move to one of the eight possible squares $(i \pm 1, j \pm 2)$ and $(i \pm 2, j \pm 1)$, if these squares are on the board.

We first became interested in knight's tour when we wanted some examples to test the behavior of a heuristic algorithm on graphs that had Hamiltonian paths. A graph is a pair (V, E), where V is a finite set of objects called vertices or nodes, and E, the set of edges, is a subset of $V \times V$ such that if $(i, j) \in E$, then $(j, i) \in E$. A Hamiltonian path (named after the famous Irish mathematician William Rowan Hamilton), is a sequence of vertices $v_1, v_2, \ldots, v_N$ that includes each vertex once, and such that $(v_i, v_{i+1}) \in E$. The path is a Hamiltonian circuit if (v_N, v_1) is also in E. Hamilton demonstrated that the dodecahedron has a Hamiltonian circuit. It is suspected [2] and [4] that determining whether or not a given graph has a Hamiltonian circuit or path is difficult in the sense that there might be no easier way than looking at all the $N!$ permutations of the vertex set and testing each for the circuit or path property.

Thus it is of some interest to have a large class of graphs that have the Hamiltonian path or circuit property. The knight's problem is: For what (n, m) does the graph derived from the $n \times m$ chessboard by the allowed knight's moves have a Hamiltonian path or circuit? Restated, the problem is: Does the $n \times m$ chessboard have a knight's path or a knight's circuit?

It is easy to show that some chessboards do not have a knight's circuit. Let us recall that the chessboard has squares of two colors, usually red and black, such that two squares that have a side in common are of different color. This implies that the knight must move in one step from one color to the other color. Thus if the board has an odd number of squares, a knight's circuit is impossible, since there are more squares of one color than the other color. But this does not rule out the possibility of a knight's path.

When we needed some examples of graphs with Hamiltonian paths, we used $n \times n$ chessboards with $n \geq 5$. We assumed that it was well known that all such boards have the required paths. But when we were asked to produce a reference, we had none. A search of the literature was called for. The standard books on recreational mathematics were little help. Kraitchik [6] had a diagram which proved that if $n \equiv 1 \pmod 4$, then the $n \times n$ chessboard has a knight's path. Ball [1] had a technique that he claimed would show that if $n \equiv 0 \pmod 4$, then the $n \times n$ chessboard has a knight's circuit. But we must confess that we were unable to fill in the details and we have doubts that the technique works. Dudeney [3] boldly states that if $n \geq 5$, then the $n \times n$ board has a knight's path, and if n is even there is a knight's circuit. Unfortunately, he neither gives a proof nor gives a reference to a proof. We were delighted to find that Kraitchik [5] had written a monograph on the knight's problem. But when we obtained a copy we were disappointed to find no proof of the general statement. Instead, there is a large collection of paths and circuits with various degrees of symmetry, the diagram for the case $n \equiv 1 \pmod 4$, and a detailed discussion of $4 \times n$ boards.

Unable to find a proof in the literature, we were forced to construct our own. The proof that follows may be of some interest to others.

In what follows, it is often necessary to refer to a particular square on a board. We may do so in either of two ways. We can refer to a square by a pair of integers (i, j) with $1 \leq i \leq n$ and $1 \leq j \leq m$ for an $n \times m$ board. The square (1,1) will be in the lower left-hand corner of the board. If we have displayed a knight's path on a board, we may instead refer to a square by a single integer i, such that square i is the ith square visited on a particular path. This second method is used throughout the figures.

Lemma 1: The $5 \times m$ board with $m \geq 5$ has a knight's path that starts in the lower left and exits at either the lower right or at the upper right.

Proof: The lemma is true for $5 \times r$ boards with $r \in \{5, 6, 7, 8, 9\}$, as shown in Figure 1. For any $m \geq 10$, partition the board into a sequence of 5×5 boards ending with a $5 \times r$ board. Clearly, we can take the knight's path for the first 5×5 board, starting in the lower left and exiting into the lower left of the next 5×5 board, and continue doing so until we land in the lower left of the $5 \times r$ board. But then we can take the knight's path for the $5 \times r$ board that ends in either the upper right or the lower right, and the lemma is proved.

Fig. 1

3	22	13	16	5
12	17	4	21	14
23	2	15	6	9
18	11	8	25	20
1	24	19	10	7

5 x 5

7	12	15	20	5
16	21	6	25	14
11	8	13	4	19
22	17	2	9	24
1	10	23	18	3

5 x 5

7	14	21	28	5	12
22	27	6	13	20	29
15	8	17	24	11	4
26	23	2	9	30	19
1	16	25	18	3	10

5 x 6

3	10	29	20	5	12
28	19	4	11	30	21
9	2	17	24	13	6
18	27	8	15	22	25
1	16	23	26	7	14

5 x 6

3	12	21	30	5	14	23
20	29	4	13	22	31	6
11	2	19	32	7	24	15
28	33	10	17	26	35	8
1	18	27	34	9	16	25

5 x 7

3	32	11	34	5	26	13
10	19	4	25	12	35	6
31	2	33	20	23	14	27
18	9	24	29	16	7	22
1	30	17	8	21	28	15

5 x 7

3	12	37	26	5	14	17	28
34	23	4	13	36	27	6	15
11	2	35	38	25	16	29	18
22	33	24	9	20	31	40	7
1	10	21	32	39	8	19	30

5 x 8

33	8	17	38	35	6	15	24
18	37	34	7	16	25	40	5
9	32	29	36	39	14	23	26
30	19	2	11	28	21	4	13
1	10	31	20	3	12	27	22

5 x 8

9	4	11	16	23	42	33	36	25
12	17	8	3	32	37	24	41	34
5	10	15	20	43	22	35	26	29
18	13	2	7	38	31	28	45	40
1	6	19	14	21	44	39	30	27

5 x 9

9	4	11	16	27	32	35	40	25
12	17	8	3	36	41	26	45	34
5	10	15	20	31	28	33	24	39
18	13	2	7	42	37	22	29	44
1	6	19	14	21	30	43	38	23

5 x 9

Lemma 2: Every $6 \times m$ board with $m \geq 5$ has a knight's circuit and a knight's path starting in the lower left and exiting at the upper left.

Proof: Looking at the 6×5 board in Figure 2, we can see that we can connect two 6×5 boards together so that we start in square 1 of the first, follow the indicated numbers until square 27, then jump to square 1 of the next board, and follow its knight's path ending in square 30, from which we can jump back to square 28 of the first board and complete its knight's path. This construction does not depend on the boards being 6×5; in fact, it will work on $6 \times k$ boards as long as the starting square is in the lower left (1, 1), the end square is diagonally below the upper left (5, 2), and the squares $(2, k-1)$ and $(4, k)$ are adjacent. From Figure 2, this is true for $6 \times r$ boards with $r \in \{5, 6, 7, 8, 9\}$. Thus every $6 \times m$ board has a knight's path beginning in the lower left and ending in the upper left, since we can partition the $6 \times m$ board into a series of 6×5 boards and one $6 \times r$ board, and connect them following the construction.

For the knight's circuit, we note that for $6 \times r$ with $r \in \{5, 6, 7, 8, 9\}$ the circuits are given in Figure 3. If $m \geq 10$, we can partition the board into a 6×5 board and a $6 \times (m - 5)$ board. Looking at the circuit for the 6×5 board in Figure 3, we can start in square 1, follow the circuit until square 28, then jump to the $6 \times (m - 5)$ board. By the half of the lemma that has already been proved, this board has a knight's path that starts at (1, 1) and ends at (5, 2), but from 28 on the 6×5 board we can jump to (5, 2), then take the path backwards until (1, 1) is reached. From (1, 1) we can jump to 29 on the 6×5 board and complete the circuit.

Fig. 2

10	19	4	29	12
3	30	11	20	5
18	9	24	13	28
25	2	17	6	21
16	23	8	27	14
1	26	15	22	7

6×5

14	23	6	28	12	21
7	36	13	22	5	27
24	15	29	35	20	11
30	8	17	26	34	4
16	25	2	32	10	19
1	31	9	18	3	33

6×6

18	23	8	39	16	25	6
9	42	17	24	7	40	15
22	19	32	41	38	5	26
33	10	21	28	31	14	37
20	29	2	35	12	27	4
1	34	11	30	3	36	13

6×7

18	31	8	35	16	33	6	45
9	48	17	32	7	46	15	26
30	19	36	47	34	27	44	5
37	10	21	28	43	40	25	14
20	29	2	39	12	23	4	41
1	38	11	22	3	42	13	24

6×8

22	45	10	53	20	47	8	35	18
11	54	21	46	9	36	19	48	7
44	23	42	37	52	49	32	17	34
41	12	25	50	27	38	29	6	31
24	43	2	39	14	51	4	33	16
1	40	13	26	3	28	15	30	5

6×9

Fig. 3

16	9	6	27	18
7	26	17	14	5
10	15	8	19	28
25	30	23	4	13
22	11	2	29	20
1	24	21	12	3

6 x 5

4	25	34	15	18	7
35	14	5	8	33	16
24	3	26	17	6	19
13	36	23	30	9	32
22	27	2	11	20	29
1	12	21	28	31	10

6 x 6

26	37	8	17	28	31	6
9	18	27	36	7	16	29
38	25	10	19	30	5	32
11	42	23	40	35	20	15
24	39	2	13	22	33	4
1	12	41	34	3	14	21

6 x 7

30	35	8	15	28	39	6	13
9	16	29	36	7	14	27	38
34	31	10	23	40	37	12	5
17	48	33	46	11	22	41	26
32	45	2	19	24	43	4	21
1	18	47	44	3	20	25	42

6 x 8

14	49	4	51	24	39	6	29	22
3	52	13	40	5	32	23	42	7
48	15	50	25	38	41	28	21	30
53	2	37	12	33	26	31	8	43
16	47	54	35	18	45	10	27	20
1	36	17	46	11	34	19	44	9

6 x 9

Lemma 3: Every $8 \times m$ board with $m \geq 5$ has a knight's path starting in the lower left and ending in the upper left, and a knight's circuit.

Proof: From Figure 4, there are $8 \times r$ circuits for $r \in \{5, 6, 7, 8, 9\}$. Since in each of these circuits the two squares $(2, r-1)$ and $(4, r)$ are adjacent, we can join two boards together to form a larger circuit. Start in square 1 and follow the circuit until one of the squares $(2, r-1)$ or $(4, r)$ is reached. Then jump to the next board at either (1, 1) or at (3, 2). Follow the circuit on this board and when it is finished jump back to the first board at the square $(2, r-1)$ or $(4, r)$, whichever has not been visited, and complete the circuit. Since this construction can be carried out for any number of boards, there is always a circuit of the $8 \times m$ board if $m \geq 5$.

The required paths for $8 \times r$, $r \in \{5, 6, 7, 8, 9\}$ are displayed in Figure 5. If $m \geq 10$, partition the board into an 8×5 board and an $8 \times (m - 5)$ board. Start in the lower left of the 8×5 board, follow the path to 34, then jump to (3, 2) of the $8 \times (m - 5)$ board. Take the circuit of the $8 \times (m - 5)$ board that ends in (1, 1) and then jump back to 35 on the 8×5 board and complete the path.

Fig. 4

26	7	28	15	24
31	16	25	6	29
8	27	30	23	14
17	32	39	34	5
38	9	18	13	22
19	40	33	4	35
10	37	2	21	12
1	20	11	36	3

8 x 5

42	21	26	5	38	13
25	4	41	12	27	6
20	43	22	37	14	39
3	24	11	40	7	28
44	19	46	23	36	15
47	2	33	10	29	8
18	45	48	31	16	35
1	32	17	34	9	30

8 x 6

22	27	40	5	38	29	14
41	4	23	28	13	6	37
26	21	12	39	50	15	30
3	42	51	24	31	36	7
20	25	32	11	52	49	16
55	2	43	46	33	8	35
44	19	56	53	10	17	48
1	54	45	18	47	34	9

8 x 7

48	13	30	9	56	45	28	7
31	10	47	50	29	8	57	44
14	49	12	55	46	59	6	27
11	32	37	60	51	54	43	58
36	15	52	63	38	61	26	5
33	64	35	18	53	40	23	42
16	19	2	39	62	21	4	25
1	34	17	20	3	24	41	22

8 x 8

42	19	38	5	36	21	34	7	60
39	4	41	20	63	6	59	22	33
18	43	70	37	58	35	68	61	8
3	40	49	64	69	62	57	32	23
50	17	44	71	48	67	54	9	56
45	2	65	14	27	12	29	24	31
16	51	72	47	66	53	26	55	10
1	46	15	52	13	28	11	30	25

8 x 9

Fig. 5

28	7	22	39	26
23	40	27	6	21
8	29	38	25	14
37	24	15	20	5
16	9	30	13	34
31	36	33	4	19
10	17	2	35	12
1	32	11	18	3

8 x 5

42	11	26	9	34	13
25	48	43	12	27	8
44	41	10	33	14	35
47	24	45	20	7	28
40	19	32	3	36	15
23	46	21	6	29	4
18	39	2	31	16	37
1	22	17	38	5	30

8 x 6

38	19	6	55	46	21	8
5	56	39	20	7	54	45
18	37	4	47	34	9	22
3	48	35	40	53	44	33
36	17	52	49	32	23	10
51	2	29	14	41	26	43
16	13	50	31	28	11	24
1	30	15	12	25	42	27

8 x 7

Fig. 5—continued

24	11	37	9	26	21	39	7
36	64	25	22	38	8	27	20
12	23	10	53	58	49	6	28
63	35	61	50	55	52	19	40
46	13	54	57	48	59	29	5
34	62	47	60	51	56	41	18
14	45	2	32	16	43	4	30
1	33	15	44	3	31	17	42

8 x 8

32	47	6	71	30	45	8	43	26
5	72	31	46	7	70	27	22	9
48	33	4	29	64	23	44	25	42
3	60	35	62	69	28	41	10	21
34	49	68	65	36	63	24	55	40
59	2	61	16	67	56	37	20	11
50	15	66	57	52	13	18	39	54
1	58	51	14	17	38	53	12	19

8 x 9

Lemma 4: Every $n \times m$ board with n odd, $\min(n, m) \geq 5$, has a knight's path that starts in the upper left and exits at the upper right.

Proof: If $m \geq 10$, partition the board into an $n \times 5$ board and an $n \times (m - 5)$ board. If the lemma holds for each of these subboards, then it holds for the whole board. Thus the result holds by induction if we can show that it holds for all $n \times r$ boards with $r \in \{5, 6, 7, 8, 9\}$ and n odd. The cases $r = 6$ and $r = 8$ have been proved in the previous two lemmas.

For the $n \times 5$ case, we have as the base of an induction the boards that appear in Figure 6. If $n \geq 10$, we partition the board into a 5×5 board and an $(n - 5) \times 5$ board. Notice that in Figure 6 the squares 16 and 17 of the 5×5 board would command the squares (3, 2) and (1, 1), respectively, of the $(n - 5) \times 5$ board. If that board has a circuit, we could go from 1 to 16 on the 5×5 board, jump to (3, 2) on the $(n - 5) \times 5$ board, take the circuit ending in (1, 1), jump back to 17 in the 5×5 board and complete the path. If $n - 5 = 6$ or $n - 5 = 8$, we have shown that the required circuit exists in the previous two lemmas.

To show that there is a circuit if $n - 5 \geq 10$, we can again partition the $(n - 5) \times 5$ board into a 5×5 board and an $(n - 10) \times 5$ board. We know from Lemma 1 that there is a knight's path on the 5×5 board that starts at the lower left and exits at the lower right. To complete the circuit we need the $(n - 10) \times 5$ board to have a knight's path that starts at the upper right and exits at the upper left, i.e., into the starting square of the path on the 5×5 board. But interchanging left and right, this is what we are trying to prove. Thus we conclude by induction that the $5 \times n$ board has the required path.

The argument for the $7 \times n$ and $9 \times n$ boards is similar. We need as our induction base the 7×5 and 9×5 boards of Figure 1 and the 7×7, 7×9, 9×7, and 9×9 boards of Figure 7.

Fig. 6

7	20	9	14	5
10	25	6	21	16
19	8	15	4	13
24	11	2	17	22
1	18	23	12	3

5 × 5

17	14	25	6	19	8	29
26	35	18	15	28	5	20
13	16	27	24	7	30	9
34	23	2	11	32	21	4
1	12	33	22	3	10	31

5 × 7

7	12	37	42	5	18	23	32	27
38	45	6	11	36	31	26	19	24
13	8	43	4	41	22	17	28	33
44	39	2	15	10	35	30	25	20
1	14	9	40	3	16	21	34	29

5 × 9

Fig. 7

9	30	19	42	7	32	17
20	49	8	31	18	43	6
29	10	41	36	39	16	33
48	21	38	27	34	5	44
11	28	35	40	37	26	15
22	47	2	13	24	45	4
1	12	23	46	3	14	25

7 × 7

13	26	39	52	11	24	37	50	9
40	81	12	25	38	51	10	23	36
27	14	53	58	63	68	73	8	49
80	41	64	67	72	57	62	35	22
15	28	59	54	65	74	69	48	7
42	79	66	71	76	61	56	21	34
29	16	77	60	55	70	75	6	47
78	43	2	31	18	45	4	33	20
1	30	17	44	3	32	19	46	5

9 × 9

Fig. 7—continued

5	20	53	48	7	22	31
52	63	6	21	32	55	8
19	4	49	54	47	30	23
62	51	46	33	56	9	58
3	18	61	50	59	24	29
14	43	34	45	28	57	10
17	2	15	60	35	38	25
42	13	44	27	40	11	36
1	16	41	12	37	26	39

9 x 7

59	4	17	50	37	6	19	30	39
16	63	58	5	18	51	38	7	20
3	60	49	36	57	42	29	40	31
48	15	62	43	52	35	56	21	8
61	2	13	26	45	28	41	32	55
14	47	44	11	24	53	34	9	22
1	12	25	46	27	10	23	54	33

7 x 9

Theorem 1: An $n \times m$ board with nm even, $\min(n, m) \geq 5$, has a knight's circuit.

Proof: If n even, $n \geq 10$, partition the board into a $5 \times m$ board and an $(n - 5) \times m$ board. Choose as the starting square the upper left-hand corner of the $(n - 5) \times m$ board. Since $n - 5$ is odd, we know from Lemma 4 that there is a knight's path on this board that will end in a square accessible to the lower right-hand corner of the $5 \times m$ board. From Lemma 1 we know that there is a knight's path of the $5 \times m$ board that starts in the indicated corner and exits at the lower left. But this was the starting square for the $(n - 5) \times m$ board, so we have constructed a knight's circuit. Of course, the same construction works if m is even and $m \geq 10$, by switching rows and columns. The only other cases are $n = 6$ or $n = 8$, and we have demonstrated in Lemma 2 and Lemma 3 how to build circuits in these cases.

Theorem 2: Every $n \times m$ board with $\min(n, m) \geq 5$ has a knight's path.

Proof: By Lemma 4 this follows if the board has n or m odd. If n or m is even the previous theorem assures a circuit and thus a path.

REFERENCES

[1] W. W. R. Ball, *Mathematical Recreations and Essays* (New York: Macmillan, 1962).

[2] S. A. Cook, "The Complexity of Theorem-Proving Procedures," *Proc. 3rd ACM Symposium on Theory of Computing*, 1971, pp. 151-158.

[3] Henry E. Dudeney, *Amusements in Mathematics* (New York: Dover, 1958).

[4] R. M. Karp, "Reducibility among Combinatorial Problems," in *Complexity of Computer Computation*, ed. by R. E. Miller & J. W. Thatcher (New York: Plenum Press, 1972), pp. 85-104.

[5] M. Kraitchik, *Le Problem du Cavalier* (Paris: Gauthiers-Villars et Cie, 1927).
[6] M. Kraitchik, *Mathematical Recreations* (New York: Dover, 1953).

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A FAMILY OF TRIDIAGONAL MATRICES

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Consider the sequence of tridiagonal determinants $\left\{P_n^{(k)}(a, b, c)\right\}_{n=1}^{\infty}$ defined by $P_n^{(k)}(a, b, c) = P_n^{(k)} = \left|(a_{ij})\right|$ where

$$a_{ij} = \begin{cases} a, & i = j \\ b, & i = j - k \\ c, & i = j + k \\ 0, & \text{otherwise} \end{cases}.$$

We shall assume $P_n^{(1)} \neq 0$. The determinant $P_n^{(k)}$ has a's down the main diagonal, b's down the diagonal k positions to the right of the main diagonal and c's down the diagonal k positions below the main diagonal.

In [1], the authors discuss $\left\{P_n^{(2)}\right\}_{n=1}^{\infty}$ and find its generating function. This note deals with a relationship that exists between

$$\left\{P_n^{(k)}\right\}_{n=1}^{\infty} \text{ and } \left\{P_n^{(1)}\right\}_{n=1}^{\infty} \text{ for } k \geq 2.$$

The first few terms of $\left\{P_n^{(1)}\right\}_{n=}^{\infty}$ with $P_0^{(1)}$ defined as one are:

$$\begin{aligned}
P_0^{(1)} &= 1 \\
P_1^{(1)} &= a \\
P_2^{(1)} &= a^2 - bc \\
P_3^{(1)} &= a^3 - 2abc \\
P_4^{(1)} &= a^4 - 3a^2bc + b^2c^2 \\
P_5^{(1)} &= a^5 - 4a^3bc + 3ab^2c^2 \\
P_6^{(1)} &= a^6 - 5a^4bc + 6a^2b^2c^2 - b^3c^3 \\
P_7^{(1)} &= a^7 - 6a^5bc + 10a^3b^2c^2 - 4ab^3c^3 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$