ON THE DENSITY OF THE IMAGE SETS OF CERTAIN ARITHMETIC FUNCTIONS—I

ROSALIND GUARALDO
St. Francis College, Brooklyn, NY 11201

1. INTRODUCTION

Throughout what follows, we will let n denote an arbitrary nonnegative integer, $S(n)$ a nonnegative integer-valued function of n, and $T(n) = n + S(n)$. We also let $\mathcal{R} = \{x | x = T(n) \text{ for some } n\}$ and $\mathbf{C}$ = complement of $\mathcal{R} = \{n \geq 0 | n \notin \mathcal{R}\}$.

It is of interest to ask whether or not the set $\mathbf{C}$ is infinite. We can also pose the question: does the set $\mathcal{R}$ have asymptotic density and, if so, does $\mathcal{R}$ (or C) have positive density? It might be suspected that if $S(n)$ is "small" there is a good chance that $\mathcal{R}$ has density. However, this suspicion is incorrect, as can be seen from the following example: for a given $n \geq 1$, let k be the unique integer satisfying $k! \leq n \leq (k + 1)! - 1$ and define

$$S(n) = \begin{cases} 0 \text{ if } k \text{ is odd} \\ 1 \text{ if } n = k! + k_1,\ k \text{ and } k_1 \text{ even},\ 0 \leq k_1 \leq (k+1)! - 1 \\ 0 \text{ if } n = k! + k_1,\ k \text{ even},\ k_1 \text{ odd and as above} \end{cases}$$

Then n or $n+1$ belongs to $\mathcal{R}$ for every natural number n, so if δ and Δ denote the lower and upper density of $\mathcal{R}$, respectively, we have $\frac{1}{2} \leq \delta \leq \Delta \leq 1$. Now if $D(n) = \{x \leq n | x = T(y) \text{ for some } y\}$ then

$$\frac{D\big((k+1)! - 1\big)}{(k+1)! - 1} = \frac{\frac{1}{2}\big((k+1)! - k!\big) - \big(k! - 1 - (k-1)!\big) - \cdots}{(k+1)! - 1} \leq \tfrac{1}{2} + o(1)$$

if k is even, and

$$\frac{D\big((k+1)! - 1\big)}{(k+1)! - 1} = \frac{(k+1)! - k! - \frac{1}{2}\big(k! - 1 - (k-1)!\big) - \cdots}{(k+1)! - 1} \geq 1 + o(1)$$

if k is odd. Hence, $\delta = \frac{1}{2}$ and $\Delta = 1$. Therefore, even if $S(n)$ can take on only the values 0 and 1, it is possible for $\mathcal{R}$ not to have density.

Let $b \geq 2$ be arbitrary and let $n = \sum_{j=0}^{k} d_j b^j$ be the unique representation of n in base b. Define $S(n) = \sum_{j=0}^{k} f(d_j, j)$, where $f(d, j)$ is a nonnegative integer-valued function of the digit d and the place where the digit occurs, and $T(n) = n + S(n)$. The consideration of functions of this form is motivated by the problem (which was posed in [1]) of showing that $\mathbf{C}$ is infinite when $T(n) = n + \sum_{j=0}^{k} d_j$. A solution, as given in [2], was obtained by recursively constructing an infinite sequence of integers in $\mathbf{C}$ for all bases b. It was also observed in [2] that if b is odd then $T(n)$ is always even. In fact, $\mathcal{R}$ is precisely the set of all nonnegative even integers when b is odd. To see this, observe that $n \equiv S(n) \pmod{b-1}$ and, therefore, $T(n) \equiv 2S(n) \pmod{b-1}$ where $S(n) = \sum_{j=0}^{k} d_j$. Hence $T(n)$ is even if b is odd. Since $T(0) = 0$, $T(n+1) \le T(n) + 2$ for every natural number n, and $T(n) \to \infty$ as $n \to \infty$, the result is proved.

2. EXISTENCE AND COMPUTABILITY OF THE DENSITY

Again, letting $n = \sum_{j=0}^{k} d_j b^j$, $S(n) = \sum_{j=0}^{k} f(d_j, j)$, and $T(n) = n + S(n)$, we prove that the density of $\mathfrak{R}$ exists and is in fact computable when suitable hypotheses are placed on the function f. We will adhere to the following notation:

$$\Omega(k, r) = \{T(x) \mid k \le x \le r\}$$
$$\Omega(r) = \Omega(0, r)$$
$$D(k, r) = |\Omega(k, r)|$$
$$D(r) = |\Omega(r)|.$$

Theorem 2.1: Let $f(d, j)$ $(d = 0, 1, \ldots, b-1)$ be a family of nonnegative integer-valued functions satisfying

(a) $f(0, j) = 0$, $j = 0, 1, 2, \ldots$
(b) $f(d, j) = o(b^j)$, $1 \le d \le b - 1$.

Then the density of $\mathfrak{R}$ exists.

Proof: First, we show that

$$D(db^k, db^k + r) = D(r), \quad 0 \le r \le b^k - 1, \quad 0 \le d \le b - 1. \tag{2.2}$$

To prove this, suppose that

$$x = db^k + \sum_{j=0}^{k-1} d_j b^j \quad \text{and} \quad y = db^k + \sum_{j=0}^{k-1} d_j' b^j.$$

Clearly $T(x) = T(y)$ if and only if

$$T\left(\sum_{j=0}^{k-1} d_j b^j\right) = T\left(\sum_{j=0}^{k-1} d_j' b^j\right).$$

Now if $d_{k-1} = d_{k-2} = \cdots = d_{k-t} = 0$ (or if $d_{k-1}' = d_{k-2}' = \cdots = d_{k-t}' = 0$), then, by assumption (a), we see that

$$T\left(\sum_{j=0}^{k-t-1} d_j b^j\right) = T\left(\sum_{j=0}^{k-1} d_j b^j\right) = T\left(\sum_{j=0}^{k-1} d_j' b^j\right).$$

We therefore have a one-one correspondence between the elements of $\Omega(db^k, db^k + r)$ and $\Omega(r)$, $0 \le r \le b^k - 1$, from which (2.2) follows. In particular, if $r = b^k - 1$, we have

$$D(db^k, (d+1)b^k - 1) = D(b^k - 1). \tag{2.3}$$

Our next lemma will enable us to relate $D(b^{k+1} - 1)$ to

$$\sum_{d=0}^{b-1} D(db^k,\ (d+1)b^k - 1).$$

Lemma 2.4: There exists an integer k_0 such that for all $k \geq k_0$ the sets $\Omega(0,\ b^k - 1),\ \Omega(b^k,\ 2b^k - 1),\ \ldots,\ \Omega((b-1)b^k,\ b^{k+1} - 1)$ are pairwise disjoint, except possibly for adjacent pairs.

Proof: The maximum value of any element in $\Omega(db^k,\ (d+1)b^k - 1)$ is at most $(d+1)b^k - 1 + M_k(k+1)$, where $M_k = \max\{f(d,\ j)\,|\,0 \leq j \leq k\}$ and the minimum value of any element in $\Omega((d+2)b^k,\ (d+3)b^k - 1)$ is at least $(d+2)b^k$. Because of assumption (b), there exists k_0' such that $f(d,\ j) < b^j/2$ for all $j \geq k_0'$ and there exists $k_0 \geq k_0'$ such that $f(d,\ j) < b^j/2 - M_{k_0'}(k_0' + 1)$, whenever $k_0 \geq k_0'$, where

$$M_{k_0'} = \max\{f(d,\ j)\,|\,0 \leq j \leq k_0'\}.$$

Therefore,

$$\sum_{j=0}^{k} f(d_j,\ j) = \sum_{j=0}^{k_0'} f(d_j,\ j) + \sum_{j=k_0'+1}^{k_0} f(d_j,\ j) + \sum_{j=k_0+1}^{k} f(d_j,\ j)$$

$$< M_{k_0'}(k_0' + 1) + \sum_{j=k_0'+1}^{k} b^j/2 - M_{k_0'}(k_0' + 1)(k - k_0)$$

$$\leq \sum_{j=k_0'+1}^{k} b^j/2 < b^k \text{ for all } k \geq k_0,$$

so, in particular, $M_k(k+1) < b^k$. Hence,

$$(d+1)b^k - 1 + M_k(k+1) < (d+2)b^k$$

whenever $k \geq k_0$, which completes the proof of the lemma.

Now $D(b^{k+1} - 1) = \sum_{d=0}^{b-1} D(db^k,\ (d+1)b^k - 1) - Q$, where Q depends on the size of the intersections of the sets

$$\Omega(0,\ b^k - 1),\ \Omega(b^k,\ 2b^k - 1),\ \ldots,\ \Omega((b-1)b^k,\ b^{k+1} - 1).$$

Define

$$\lambda_{d,k} = |\Omega(db^k,\ (d+1)b^k - 1) \cap \Omega(d+1)b^k,\ ((d+2)b^k - 1)|,\ 0 \leq d \leq b-2.$$

Using Lemma 2.4 and Equation (2.3), we obtain

$$D(b^{k+1} - 1) = bD(b^k - 1) - \sum_{d=0}^{b-1} \lambda_{d,k},\ k \geq k_0. \tag{2.5}$$

Let

$$A_k = D(b^k - 1)/b^k \text{ and } \varepsilon_k = \sum_{d=0}^{b-2} \lambda_{d,k}/b^{k+1},\ k \geq k_0.$$

Then 2.5 can be rewritten as

$$A_{k+1} - A_k = -\varepsilon_k.$$

Therefore,

$$\begin{aligned} A_{k+1} - A_k &= -\varepsilon_k \\ A_k - A_{k-1} &= -\varepsilon_{k-1} \\ &\vdots \\ A_{k_0+1} - A_{k_0} &= -\varepsilon_{k_0} \end{aligned}$$

and by telescoping, we obtain

$$A_{k+1} = A_{k_0} - \sum_{j=k_0}^{k} \varepsilon_j.$$

Replacing $k + 1$ by k yields

$$A_k = A_{k_0} - \sum_{j=k_0}^{k-1} \varepsilon_j, \; k \geq k_0. \tag{2.6}$$

Obviously, $1/b^k \leq A_k \leq 1$ and $\sum_{j=k_0}^{k-1} \varepsilon_j = A_{k_0} - A_k < A_{k_0} \leq 1$. Thus $\sum_{j=k_0}^{k} \varepsilon_j$ is a series of nonnegative terms bounded above by A_{k_0}, hence is convergent. Let

$$L = A_{k_0} - \sum_{j=k_0}^{\infty} \varepsilon_j \tag{2.7}$$

(We have just shown that $0 \leq L \leq 1$). Then, (2.6) yields

$$A_k = L + \sum_{j=k}^{\infty} \varepsilon_j, \; k \geq k_0;$$

i.e.,

$$A_k = L + o(1). \tag{2.8}$$

Hence

$$D(b^k - 1) = Lb^k + o(b^k). \tag{2.9}$$

Using (2.3), (2.4), (2.9), and recalling the definition of the $\lambda_{d,k}$ and the ε_k, we have

$$\begin{aligned} D(db^k - 1) &= \sum_{c=0}^{d-1} D(cb^k, \, (c+1)b^k - 1) - \sum_{c=0}^{d-2} \lambda_{d,k} \\ &= \sum_{c=0}^{d-1} (Lb^k + o(b^k)) + O(b^{k+1}\varepsilon_k) = db^k L + o(b^k); \end{aligned}$$

i.e.,

$$D(db^k - 1) = db^k L + o(b^k). \tag{2.10}$$

Now let $n = \sum_{j=0}^{k} d_j b^j$ be any nonnegative integer. Then

$$\begin{aligned} D(n) &= D\left(\sum_{j=0}^{k} d_j b^j\right) \\ &= D(d_k b^k - 1) + D\left(d_k b^k, \, \sum_{j=0}^{k} d_j b^j\right) - Q, \end{aligned}$$

where Q is the number of elements that the sets

$$\Omega(d_k b^k - 1) \quad \text{and} \quad \Omega\left(d_k b^k, \sum_{j=0}^{k} d_j b^j\right)$$

have in common. Therefore, if n is sufficiently large, then by using (2.10), (2.2), and the definition of the $\lambda_{d,k}$, we have

$$D(n) = d_k b^k L + o(b^k) + D\left(\sum_{j=0}^{k-1} d_j b^j\right) + o(b^k) = d_k b^k L + D\left(\sum_{j=0}^{k-1} d_j b^j\right) + o(b^k).$$

Applying the same reasoning to the quantities $D\left(\sum_{j=0}^{t} d_j b^j\right)$, $k_0 \le t \le k - 1$, we eventually obtain

$$D(n) = L\left(\sum_{j=k_0}^{k} d_j b^j\right) + D\left(\sum_{j=0}^{k_0-1} d_j b^j\right) + \sum_{j=k_0}^{k} o(b^j);$$

i.e.,

$$D(n) = L\left(n - \sum_{j=0}^{k_0-1} d_j b^j\right) + D\left(\sum_{j=0}^{k_0-1} d_j b^j\right) + o(n).$$

Dividing both sides of this equation by n yields

$$D(n)/n = L + o(1),$$

which proves the density of $\mathcal{R}$ is L.

Remark: It should be noted that Equation (2.2), and therefore the above proof of Theorem 2.2, breaks down if we lift the condition $f(0, j) = 0$.

A particular case of Theorem 2.1 of interest occurs when we assume that f depends only on d:

Corollary 2.11: If $f(d)$ is an arbitrary nonnegative function of d, $1 \le d \le b - 1$, and $f(0) = 0$, then the density of $\mathcal{R}$ exists and is equal to L, where L is defined as in Equation (2.7).

We also easily obtain the following two corollaries to Theorem 2.1:

Corollary 2.12: $L < 1$ if and only if the function $T(n)$ is not one-one.

Proof: We have

$$L = A_{k_0} - \sum_{j=k_0}^{\infty} \varepsilon_j = A_k - \sum_{j=k}^{\infty} \varepsilon_j, \text{ for all } k \ge k_0,$$

where k_0 is defined as in Lemma 2.4. If $T(x) = T(y)$, $x \ne y$, and k is such that $k \ge k_0$ and $x \le b^k - 1$, $y \le b^k - 1$, then, since $A_k = D(b^k - 1)/b^k$, it follows that $L \le A_k < 1$. If T is one-one, then it follows from the definition of the A_k and the ε_k that $A_k = 1$ and $\varepsilon_k = 0$ for all k, so $L = 1$.

Corollary 2.13: If $f(d, j) = f(d)$ depends only on d and if $f(0) = 0$ and $f(b - 1) \ne 0$, then $L < 1$.

Proof: Let $f(b - 1) = s > 0$. Then $T(b^k - 1) = T\big((b-1)b^{k-1} + (b-1)b^{k-2} + \cdots + b - 1\big) = b^k - 1 + ks$.

Now, if k is such that $ks - 1 - f(1) < b^k$ and $n = \sum_{j=0}^{r} d_j b^j$ satisfies $T(n) = ks - 1 - f(1)$, then $n < b^k$ since $T(n) \geq n$. Hence $T(b^k + n) = T(b^k) + T(n) = b^k + f(1) + ks - 1 - f(1) = b^k - 1 + ks = T(b^k - 1)$. Therefore T is not one-one, so $L < 1$ by the above corollary. If there is never any n which satisfies the equation $T(n) = ks - 1 - f(1)$, then almost all integers of the form $ks - 1 - f(1)$, $k = 1, 2, 3, \ldots$, do not belong to $\mathfrak{R}$, hence, $\mathcal{C}$ has positive density, so $L < 1$ in this case also.

Remark: The problem posed in [1] is now an immediate consequence of the above corollary.

More generally, it seems to be true that if $f(d)$ is not identically 0 and $f(0) = 0$, then we again have $L < 1$. We let this statement stand as a conjecture. Note that the hypothesis $f(0) = 0$ is essential; for example, if f is any nonzero constant, then $T(n)$ is strictly increasing and therefore $L = 1$.

There is another question which can be raised about the value of the density L: must one always have $L > 0$ under the hypotheses of Theorem 2.1? Again, the proof of this result, if true, seems to be elusive. Since

$$L = A_k - \sum_{j=k}^{\infty} \varepsilon_j \text{ for } k \geq k_0,$$

we see that $L = 0$ if and only if $A_k = o(1)$, which means that the function $T(n)$ must be very far from being one-one.

3. EXISTENCE OF THE DENSITY WHEN $f(d, j) = 0(b^j/j^2 \log^2 j)$

The main drawback to Theorem 2.1 is the condition $f(0, j) = 0$. It seems to be difficult to prove that the density of $\mathfrak{R}$ exists if we assume only that $f(d, j) = o(b^j)$ for all digits d. On the other hand, it also seems to be difficult to find an example of an image set $\mathfrak{R}$ which does not have density under the latter assumption on f, so that the statement that $\mathfrak{R}$ does have density under this assumption will be left as a conjecture. However, the following weaker result does hold:

Theorem 3.1: If $f(d, j) = 0(b_j/j^2 \log^2 j)$ for all d, then the density of $\mathfrak{R}$ exists.

Proof: Letting $n = \sum_{j=0}^{k} d_j b^j$, we have

$$S(n) = \sum_{j=0}^{k} 0(b^j/j^2 \log^2 j) = 0(b^k/k^2 \log^2 k). \tag{3.2}$$

Now if $r \leq s \leq t$ ($r < t$) and $s < b^{k+1}$, then, letting D and Ω be the same as in the proof of Theorem 2.1, we see that

$$D(r, t) = D(r, s) + D(s + 1, t) - |\Omega(r, s) \cap \Omega(s + 1, t)|.$$

Hence, by (3.2),

$$D(r, t) = D(r, s) + D(s + 1, t) + 0(b^k/k^2 \log^2 k). \tag{3.3}$$

In particulr, if $r = 0$, $s = b^{k-1} - 1$, and $t = b^k - 1$, then

$$D(b^k - 1) = D(0,\ b^{k-1} - 1) + D(b^{k-1},\ b^k - 1) + 0\big(b^{k-1}/(k - 1)^2 \log^2(k - 1)\big).$$

Similarly, we see that

$$D(b^q - 1) = D(0,\ b^{q-1} - 1) + D(b^{q-1},\ b^q - 1) + 0\big(b^{q-1}/(q - 1)^2 \log^2(q - 1)\big),$$
$$1 \le q \le k - 1.$$

Using the two latter equations and (3.2), we obtain

$$D(b^k - 1) = D(0) + D(1,\ b - 1) + \cdots + D(b^{q-1},\ b^q - 1) \qquad (3.4)$$
$$+ \cdots + D(b^{k-1},\ b^k - 1) + 0(b^k/k^2 \log^2 k).$$

Let us now consider the quantity $D\big(db^k,\ (d + 1)b^k - 1\big)$. From (3.3), we have

$$D\big(db^k,\ (d + 1)b^k - 1\big) = D(db^k,\ db^k) + D(db^k + 1,\ db^k + b - 1)$$
$$+ D\big(db^k + b,\ (d + 1)b^k - 1\big) + 0(b^k/k^2 \log^2 k).$$

A second application of (3.3) yields

$$D\big(db^k,\ (d + 1)b^k - 1\big) = D(db^k,\ db^k) + D(db^k + 1,\ db^k + b - 1)$$
$$+ D(db^k + b,\ db^k + b^2 - 1) + D(db^k + b^2,\ (d + 1)b^k - 1)$$
$$+ 0(b^k/k^2 \log^2 k),$$

and by repeatedly applying (3.3), we eventually obtain

$$D\big(db^k,\ (d + 1)b^k - 1\big) = D(db^k,\ db^k) + D(db^k + 1,\ db^k + b - 1) \qquad (3.5)$$
$$+ \cdots + D(db^k + b^q,\ db^k + b^{q+1} - 1)$$
$$+ \cdots + D(db^k + b^{k-1},\ db^k + b^k - 1) + 0(b^k/k \log^2 k).$$

Since all integers x satisfying

$$db^k + b^q \le x \le db^k + b^{q+1} - 1 \quad (0 \le q \le k - 1)$$

have the same number of leading zeros, there is a one-one correspondence between the elements of $\Omega(db^k + b^q,\ db^k + b^{q+1} - 1)$ and $\Omega(b^q,\ b^{q+1} - 1)$, i.e.,

$$D(db^k + b^q,\ db^k + b^{q+1} - 1) = D(b^q,\ b^{q+1} - 1).$$

Using this fact, (3.5) becomes

$$D\big(db^k,\ (d + 1)b^k - 1\big) = D(0) + D(1,\ b - 1) \qquad (3.6)$$
$$+ \cdots + D(b^{k-1},\ b^k - 1) + 0(b^k/k \log^2 k),$$

and (3.4) and (3.6) imply that

$$D\big(db^k,\ (d + 1)b^k - 1\big) = D(b^k - 1) + 0(b^k/k \log^2 k). \qquad (3.7)$$

Now, from (3.7),

$$D(b^{k+1} - 1) = D(b^k - 1) + D(b^k,\ b^{k+1} - 1) + 0(b^k/k^2 \log^2 k)$$
$$= D(b^k - 1) + D(b^k,\ 2b^k - 1) + D(2b^k,\ b^{k+1} - 1)$$
$$+ 0(b^k/k^2 \log^2 k)$$
$$= 2D(b^k - 1) + D(2b^k,\ b^{k+1} - 1) + 0(b^k/k \log^2 k).$$

By repeated application of (3.7), we have

$$D(b^{k+1}-1) = bD(b^k-1) + 0(b^k/k\log^2 k). \tag{3.8}$$

Letting $A_k = D(b^k-1)/b^k$, (3.8) becomes

$$b^{k+1}A_{k+1} - b^{k+1}A_k = 0(b^k/k\log^2 k)$$

and therefore

$$A_{k+1} - A_k = 0(1/k\log^2 k).$$

Since $\sum_{j=0}^{k} 0(1/j\log^2 j) = 0(1/\log k)$, there exists a constant L such that

$$A_k = L + 0(1/\log k). \tag{3.9}$$

Let $n = d_{k_1}b^{k_1} + d_{k_2}b^{k_2} + \cdots$ be any integer, each $d_{k_j} \neq 0$. Then

$$D(n) = D(d_{k_1}b^{k_1}-1) + D(d_{k_1}b^{k_1}, n) + 0(b^{k_1}/k_1^2\log^2 k_1).$$

By the same reasoning used to obtain (3.8), we see that

$$D(d_{k_1}b^{k_1}-1) = d_{k_1}D(b^{k_1}-1) + 0(b^{k_1}/k_1\log^2 k_1).$$

Therefore, by (3.9), we have

$$D(n) = d_{k_1}b^{k_1}(L + 0(1/\log k_1)) + 0(b^{k_1}/k_1\log^2 k_1)$$
$$+ D(d_{k_1}b^{k_1}, d_{k_1}b^{k_1} + d_{k_2}b^{k_2} + \cdots).$$

Since $d_{k_j} \neq 0$ for any j, we know that

$$D(d_{k_1}b^{k_1}, d_{k_1}b^{k_1} + d_{k_2}b^{k_2} + \cdots) = D(d_{k_2}b^{k_2} + \cdots)$$

[c.f. the reasoning applied between equations (3.5) and (3.6)]. Hence,

$$D(n) = d_{k_1}b^{k_1}(L + 0(1/\log k_1)) + 0(b^{k_1}/k_1\log^2 k_1) + D(d_{k_2}b^{k_2} + \cdots).$$

Continuing in this manner, we have

$$D(n) = nL + 0(b^{k_1}/k_1\log^2 k_1) + \sum_{j=0}^{k_1} 0(b^j/\log j) = nL + 0(b^{k_1}/\log k_1).$$

This last equation shows that the density of $\mathfrak{R}$ is L, q.e.d.

Remark I: This theorem, in contrast to Theorem 2.1, has the drawback that no formula for the density of $\mathfrak{R}$ has been derived.

Remark II: It is interesting to note that there exist sets $\mathfrak{R}$ which do not have density under the assumption that $f(d, j) = 0(b^j)$. For example, let $f(d, j) = 0$ if j is even and $f(d, j) = b^j$ if j is odd. Evidently,

$$T\left(b^k + \sum_{j=0}^{k-1} d_j b^j\right) = b^k + \sum_{j=0}^{k-1} d_j b^j + b^k + b^{k-2} + \cdots + b \geq 2b^k$$

if k is odd, and

$$T\left(b^k + \sum_{j=0}^{k-1} d_j b^j\right) = b^k + \sum_{j=0}^{k-1} d_j b^j + b^{k-1} + b^{k-3} + \cdots + b$$

if k is even.

Therefore, the number of integers between b^k and $2b^k$ in $\mathcal{P}$ if k is odd is at most $1 + b^{k-2} + b^{k-4} + \cdots + b$, and the number of integers between b^k and $2b^k$ in $\mathcal{P}$ if k is even is at least $b^k - b^{k-1} - b^{k-3} - \cdots - b$. Hence, if we let δ and Δ denote the lower and upper density of $\mathcal{P}$, respectively, we see that

$$\delta \leq 1/b^2 + 1/b^4 + 1/b^6 + \cdots = 1/(b^2 - 1)$$

and

$$\Delta \geq 1 - 1/b - 1/b^3 - 1/b^5 - \cdots = 1 - b/(b^2 - 1).$$

Since $1 - b/(b^2 - 1) > 1/(b^2 - 1)$ when $b > 2$, it follows that $\mathcal{P}$ does not have density if $b \neq 2$.

It is also interesting that we can obtain examples in which the set $\mathcal{P}$ is of density 0 if $f(d, j) = 0(b^j)$. For example, if $b = 10$ and $f(d, j) = 0$ if $d \neq 1$ and $f(d, j) = 8 \cdot 10^j$ if $d = 1$, then no member of $\mathcal{P}$ has a 1 anywhere in its decimal representation, and the set

$$n = \left\{\sum_{j=0}^{k} d_j 10^j,\ d_j \neq 1,\ 0 \leq j \leq k\right\}$$

is a set which is well known to have density 0.

Corollary 3.10: If $f(d)$ is an arbitrary nonnegative function of the digit d, then the density of $\mathcal{P}$ exists.

ACKNOWLEDGMENT

The author wishes to thank her thesis advisor, Professor Eugene Levine, of Adelphi University, for the guidance received from him during the preparation of this work.

REFERENCES

1. "Problem E 2408," proposed by Bernardo Racomon, *American Math. Monthly*, Vol. 80, No. 4 (April 1973), p. 434.
2. "Solution to Problem E 2408," *American Math. Monthly*, Vol. 81, No. 4 (April 1974), p. 407.

*****