PROPERTIES OF GENERATING FUNCTIONS OF A CONVOLUTION ARRAY

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A sequence of sequences S_k which arise from inverses of matrices containing certain columns of Pascal's triangle provided a fruitful study reported by Hoggatt and Bicknell [1], [2], [3], [4]. The sequence $S_1 = \{1, 1, 2, 5, 14, 42, \ldots\}$ is the sequence of Catalan numbers. Convolution arrays for these sequences were computed, leading to classes of combinatorial and determinant identities and a web of inter-relationships between the sequences S_k. The inter-relationships of the generating functions of these related sequences led to the H-convolution transform of Hoggatt and Bruckman [5], which provided proof of all the earlier results taken together as well as generalizing to any convolution array. The development required computations with infinite matrices by means of the generating functions $S_k(x)$ for the columns containing the sequences S_k. In this paper, properties of the generating functions $S_k(x)$ are studied and extended.

1. INTRODUCTION

We define $S_k(x)$ as in Hoggatt and Bruckman [5]. Let $f(x)$ be the generating function for a sequence $\{f_i\}$ so that

$$f(x) = \sum_{i=0}^{\infty} f_i x^i = \sum_{i=0}^{\infty} a_{i,0} x^i \tag{1.1}$$

where $f(0) = f_0 = a_{00} \neq 0$ and

$$[f(x)]^{j+1} = \sum_{i=0}^{\infty} a_{ij} x^i, \quad j = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{1.2}$$

where $a_{i,-1} = 1$ if $i = 0$ and $a_{i,-1} = 0$ if $i \neq 0$. Form a new sequence with generating function $S_1(x)$ given by

$$S_1(x) = \sum_{i=0}^{\infty} \frac{a_{ii}}{i+1} x^i = \sum_{i=0}^{\infty} s_i x^i, \tag{1.3}$$

where $\{a_{ii}\}$ was generated in the convolution array by $f(x)$ as in (1.2). Then if we let $f(x) = S_0(x)$, from [5] we have $f(xS_1(x)) = S_1(x)$,

$$f(xS_k(x)) = S_k(x) \tag{1.4}$$

and

$$f(xS_k^k(x)) = S_k(x), \tag{1.5}$$

as well as

$$S_k^j(x) = \sum_{i=0}^{\infty} \frac{j}{ki + j} a_{i,\,ki+j-1} x^i, \quad k = 0, 1, 2, \ldots \tag{1.6}$$

In particular, if $f(x) = 1/(1 - x)$, we have the generating functions for the columns of Pascal's triangle and the sequences S_k are the Catalan and related sequences reported in [1], [2], [3], [4], and $a_{i,\,ki+j-1}$ is the binomial $\binom{(i+1)k + j - 1}{k}$. The sequence generated by $S_k^j(x)$ is the $(j - 1)$st convolution of the sequence S_k. The sequence S_k is formed by taking the absolute values of the elements of the first column of the matrix inverse of a matrix P_k, where P_k is formed by placing every $(k + 1)$st column of Pascal's triangle on and below the main diagonal, with zeroes elsewhere. P_0 is Pascal's triangle itself, and P_1 contains every other column of Pascal's triangle and gives the Catalan numbers 1, 1, 2, 5, 14, 42, ..., as the sequence S_1.

We now discuss properties of the generating functions $S_k(x)$.

2. THE GENERATING FUNCTIONS $S_k(x)$

We begin with

$$f(xS(x)) = S(x) \tag{2.1}$$

by assuming that $f(x)$ is analytic about $x = 0$ and $f(0) = 1$. We also note that $S(x) \neq 0$ for finite x, since $S(x) = 0$ would violate $f(0) = 1$.

Theorem 2.1: If $f(xS(x)) = S(x)$, then $S(x/f(x)) = f(x)$.

Proof: Note that $f(x) \neq 0$ for finite x. Let $y = xS(x)$ so that $f(y) = S(x)$ and $x = y/S(x) = y/f(y)$. Therefore, $f(y) = S(y/f(y))$. Changing to x we get $S(x/f(x)) = f(x)$.

Theorem 2.2: If $S(x/f(x)) = f(x)$, then $f(xS(x)) = S(x)$.

Proof: Let $y = x/f(x)$. Then $S(y) = f(x)$, $x = yf(x) = yS(y)$ which implies $f(yS(y)) = f(x) = S(y)$ so that $f(xS(x)) = S(x)$.

Theorem 2.3: The solution to $f(xS(x)) = S(x)$ is unique.

Proof: Assume $f(xS(x)) = S(x)$ and $f(xT(x)) = T(x)$. We shall show that $T(x) = S(x)$. By Theorem 2.1, $S(x/f(x)) = f(x)$. Let $x = xT(x)$ so that

$$S(xT(x))/f(xT(x)) = S(xT(x)/T(x)) = S(x).$$

But also

$$S(xT(x))/f(xT(x)) = f(xT(x)) = T(x).$$

Thus, $S(x) \equiv T(x)$.

Theorem 2.4: In $S(x/f(x)) = f(x)$, $f(x)$ is unique.

Proof: Assume $S(x/f(x)) = f(x)$ and $S(x/g(x)) = g(x)$. Apply Theorem 2.1, $S(x) = f(xS(x))$, letting $x = x/g(x)$. Then $S(x/g(x))$ becomes

$$S(x/g(x)) = f\left[(x/g(x))S(x/g(x))\right] = f\left[(x/g(x)) \cdot g(x)\right] = f(x),$$

but $S(x/g(x)) = g(x)$ so that $f(x) \equiv g(x)$.

3. THE GENERATING FUNCTIONS $S_k(x)$ WHERE $S_0(x)$ GENERATES PASCAL'S TRIANGLE

We now go on to another phase of this problem. Let

$$S_0(x) = \frac{1}{1-x} = f(x) \tag{3.1}$$

and $S_0(xS_1(x)) = S_1(x)$ be the unique solution, and from $S_1(x/S_0(x)) = S_1(x)$, when $x = 0$ we have $S_1(0) = S_0(0) = 1$. From

$$S_k\left(xS_{k+1}(x)\right) = S_{k+1}(x) \tag{3.2}$$

one can easily prove

$$S_0\left(xS_k^k(x)\right) = S_k(x) \tag{3.3}$$

for all integral k as in Hoggatt and Bruckman [5].

Thus from $S_0(x) = 1/(1-x)$, we have

$$S_0\left(xS_k^k(x)\right) = \frac{1}{1 - xS_k^k(x)} = S_k(x)$$

or

$$xS_k^{k+1}(x) - S_k(x) + 1 = 0, \quad k > 0,$$

and from

$$S_0\left(x/S_{-k}^k(x)\right) = \frac{1}{1 - x/S_{-k}^k(x)} = S_{-k}(x),$$

$$xS_{-k}^{-k+1}(x) - S_{-k}(x) + 1 = 0, \quad k \geq 0.$$

Clearly, $S_0\left(xS_0^0(x)\right) = S_0(x)$. Thus, uniformly

$$xS_k^{k+1}(x) - S_k(x) + 1 = 0 \tag{3.4}$$

for all integral k.

In particular, by (3.4),

$$xS_1^2(x) - S_1(x) + 1 = 0,$$

$$S_1(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

Clearly, $S_1(x)$ is undefined for $x > 1/4$. The solution with the positive radical is unbounded at the origin, while

$$S_1(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

is bounded at the origin, and $\lim\limits_{x \to \infty} S_1(x) = 1$. $S_1(x)$ is the generating function for the Catalan numbers. Note that $S_1\big(xS_2(x)\big) = S_2(x)$ leads to

$$S_2(x) = \frac{1 - \sqrt{1 - 4xS_2(x)}}{2xS_2(x)}$$

defined for $xS_2(x) < 1/4$, where $\lim\limits_{x \to \infty} xS_2(x) = 0$ while $S_2(x) \neq 0$ for any x.

We now proceed to the proof that

$$zS^k(z) - S(z) + 1 = 0$$

has only one continuous bounded function in the neighborhood of the origin. We first need a theorem given by Morris Marden [6, p. 3, Theorem (1.4)]:

Theorem: The zeroes of a polynomial are continuous functions of the coefficients.

Theorem 3.1: There is one and only one continuous solution to

$$zS^k(z) - S(z) + 1 = 0$$

which is bounded in the neighborhood of the origin, and this solution is such that $\lim\limits_{x \to 0} S(x) = 1$.

Proof: Let $S_1^*(z), S_2^*(z), \ldots, S_k^*(z)$ be the continuous zeroes (solutions) to $zS^k(z) - S(z) + 1 = 0$, and rewrite this as

$$S^k(z) - S(z)/z + 1/z = 0, \quad z \neq 0.$$

$$(S - S_1^*)(S - S_2^*) \ldots (S - S_k^*) = S^k - S/z + 1/z = 0.$$

Therefore, $S_1^*S_2^*S_3^* \ldots S_k^* = (-1)^k/z$ as the last coefficient, and

$$S_1^*S_2^*S_3^* \cdots S_k^*\left(\frac{1}{S_1^*} + \frac{1}{S_2^*} + \frac{1}{S_3^*} + \cdots + \frac{1}{S_k^*}\right) = (-1)^k/z$$

from the next-to-last coefficient. Therefore,

$$\frac{1}{S_1^*} + \frac{1}{S_2^*} + \frac{1}{S_3^*} + \cdots + \frac{1}{S^*} = 1. \tag{3.5}$$

Let $S_1^*(z)$ be bounded in the neighborhood; then

$$\lim_{z \to 0}\left(zS_1^{*k}(z) - S_1^*(z) + 1\right) = \lim_{z \to 0} zS_1^{*k}(z) - \lim_{z \to 0} S_1^*(z) + 1 = 0,$$

$\lim_{z\to 0} zS_1^{*k}(z) = 0$, and $\lim_{z\to 0} S_1^*(z) = S_1^*(0) = 1$. Thus $\lim_{z\to 0} 1/S_1^*(z) = 1$.

Suppose $S_j^*(z)$ is continuous but unbounded in the neighborhood of $z = 0$. Then $\lim_{z\to 0} 1/S_j^*(z) = 0$. From (3.5), we therefore conclude that $S_1^*(z)$ is the only continuous and bounded solution to our equation as $z \to 0$. We also note that since the right side is indeed 1 for all $z \neq 0$, there is *one* bounded solution. This concludes the proof of Theorem 3.1.

Theorem 3.2: $S_{-m}(x) = \dfrac{1}{S_{m-1}(-x)}$.

Proof: $S_{-m}(x)$ satisfies

$$xS_{-m}^{-m+1}(x) - S_{-m}(x) + 1 = 0.$$

Multiply through by $S_{-m}^{-1}(x)$ to yield

$$xS_{-m}^{-m}(x) - 1 + S_{-m}^{-1}(x) = 0.$$

Replace x by $(-x)$,

$$-xS_{-m}^{-m}(-x) - 1 + S_{-m}^{-1}(-x) = 0,$$

which can be rewritten as

$$x\left(S_{-m}^{-1}(-x)\right)^m - \left(S_{-m}^{-1}(-x)\right) + 1 = 0.$$

This is precisely the polynomial equation satisfied by $S_{m-1}(x)$, which is

$$xS_{m-1}^m(x) - S_{m-1}(x) + 1 = 0.$$

Since $S_{m-1}(0) = 1$, it is the unique continuous solution which is bounded in the neighborhood of the origin. If $S_{-m}(x)$ is such that

$$\lim_{x\to 0} S_{-m}(x) = S_{-m}(0) = 1,$$

then

$$\lim_{x\to 0} S_{-m}^{-1}(x) = S_{-m}^{-1}(0) = 1.$$

Therefore, by Theorem 3.1, we conclude that

$$S_{-m}^{-1}(-x) = S_{m-1}(x) \quad \text{or} \quad S_{-m}(x) = \frac{1}{S_{m-1}(-x)}$$

which concludes the proof of Theorem 3.2.

Theorem 3.3: If $S_k(x)$ obeys

$$xS_k^{k+1}(x) - S_k(x) + 1 = 0,$$

then $S_k(x) \neq 0$ for any finite x, $k \neq -1$.

Proof: Let $S_k(x)$ be the continuous solution as a function of x. Then,

$$\operatorname*{limit}_{x \to x_0} S_k(x) = S_k(x_0),$$

where $S_k(x_0)$ is finite. If $S_k(x_0) = 0$, then

$$\operatorname*{limit}_{x \to x_0} \left[xS_k^{k+1}(x) - S_k(x) + 1 \right] = 1 \neq 0,$$

which contradicts the fact that $xS_k^{k+1}(x) - S_k(x) + 1 = 0$. However, if $k = -1$, then $S_{-1}(x) = 1 + x$, which is zero for $x = -1$. For all other k, $S_k(x) = 0$ for all finite x.

4. EXTENDED RESULTS FOR GENERALIZED PASCAL TRIANGLES

The results of Section 3 can be extended. Let

$$f(x) = \frac{1}{1 + cxg(x)}, \quad c \neq 0, \; f(0) = 1, \tag{4.1}$$

$g(x)$ a polynomial in x. Then $f\big(xS(x)\big) = S(x)$ yields

$$\frac{1}{1 + cxS(x)g\big(xS(x)\big)} = S(x) \tag{4.2}$$

or

$$1 - S(x) + cxS(x)g\big(xS(x)\big) = 0$$

which is a polynomial in $S(x)$. Because of the 1 and $-S(x)$ relationships in the equation, all of the previous results hold. For example, all of the generalized Fibonacci numbers from the generalized Pascal triangles arising from the coefficients generated in the expansions of the multinomials $(1 + x + x^2 + \cdots + x^m)^n$ will have convolution arrays governed by the results of this paper and similar to those reported for Pascal's triangle in [1] through [4].

Now, looking at (4.2), since $g(0) = 1$, the polynomial in S is of the form

$$\frac{1}{x^k} + \frac{cx - 1}{x^k}S + \cdots + S^k(x) = 0.$$

As before, inspecting the coefficients yields, for roots $S_1^*, S_2^*, \ldots, S_k^*$,

$$S_1^*S_2^*S_3^* \ldots S_k^* = (-1)^k/x^k$$

and

$$S_1^*S_2^*S_3^* \ldots S_k^*\left(\frac{1}{S_1^*} + \frac{1}{S_2^*} + \cdots + \frac{1}{S_k^*}\right) = \frac{(cx - 1)(-1)^k}{x^k}$$

so that

$$\frac{1}{S_1^*} + \frac{1}{S_2^*} + \cdots + \frac{1}{S_k^*} = 1 - cx.$$

Now,

$$\operatorname*{limit}_{x \to 0} \left(\frac{1}{S_1^*(x)} + \frac{1}{S_2^*(x)} + \cdots + \frac{1}{S_k^*(x)} \right) = 1;$$

Thus, $\operatorname*{limit}_{x \to 0} 1/S_1^*(0) = 1$ and $\operatorname*{limit}_{x \to 0} 1/S_j^*(x) = 0$, and we again have one and only one bounded and continuous solution near the origin.

REFERENCES

1. V. E. Hoggatt, Jr., & Marjorie Bicknell, "Pascal, Catalan and General Sequence Convolution Arrays in a Matrix," *The Fibonacci Quarterly*, Vol. 14, No. 2 (April 1976), pp. 135-142.
2. V. E. Hoggatt, Jr., & Marjorie Bicknell, "Catalan and Related Sequences Arising from Inverses of Pascal's Triangle Matrices," *The Fibonacci Quarterly*, Vol. 14, No. 5 (December 1976), pp. 395-404.
3. V. E. Hoggatt, Jr., & Marjorie Bicknell, "Sequences of Matrix Inverses from Pascal, Catalan, and Related Convolution Arrays," *The Fibonacci Quarterly*, Vol. 14, No. 3 (October 1976), pp. 224-232.
4. V. E. Hoggatt, Jr., & Marjorie Bicknell, "Numerator Polynomial Coefficient Arrays for Catalan and Related Sequence Convolution Triangles," *The Fibonacci Quarterly*, Vol. 15, No. 1 (February 1977), pp. 30-34.
5. V. E. Hoggatt, Jr., & Paul S. Bruckman, "The H-Convolution Transform," *The Fibonacci Quarterly*, Vol. 13, No. 4 (December 1975), pp. 357-368.
6. Morris Marden, *The Geometry of the Zeros of a Polynomial in a Complex Variable*, Mathematical Surveys III, American Mathematical Society, 1948.

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