A NEW SERIES

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There is a series very similar to Fibonacci's that also displays some interesting properties. An article by Marjorie Bicknell [1] in *The Fibonacci Quarterly* (February 1971) casually mentions the series as a result of more investigation of Pascal's Triangle. The series is 0, 1, 1, 1, 2, 3, 4, 6, 9, 13, ... Each term is found from the relationship

$$F^*_{n+1} = F^*_n + F^*_{n-2}.$$

The series resulted from my research in the Great Pyramid of Gizeh, where the base-to-height ratio is $\pi/2$ and the slant height of a side to the height approximates $\sqrt{\phi}$. ϕ represents the series limit of the Fibonacci Series (see Figure 1).

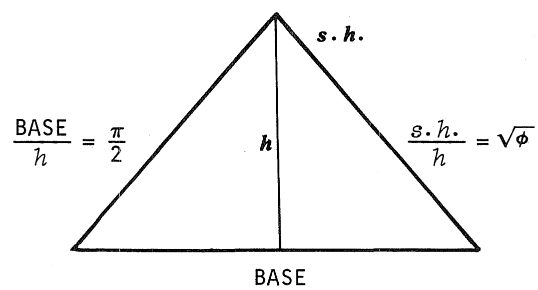


Fig. 1

One of the properties of the new series presented in this paper is that it better fits into the design features of the pyramid than does the accepted fact that Fibonacci's Series limit is intended to be decoded.

The series limit of the new series is represented by the symbol ψ and represents the number 1.46557123..., which will be used as

$$\psi = 1.465571232$$

in this paper.

Referring to Figure 1 again, the ratio of slant height to height is much better represented by the following relationship,

$$\frac{s.h.}{h} = \sqrt{\frac{1 - \ln \psi}{\ln \psi}}$$

This relationship yields a slant height which is only 1.67 inches from the measured values. Fibonacci's Series limit being employed yields a slant height that is 2.7 inches greater than the measured value; the new series limit yielding 1.67 inches less. This is not to dispute the existence of the Fibonacci Series limit as being intended, but to confirm that both expressions are intended by the Designer of the Great Pyramid.

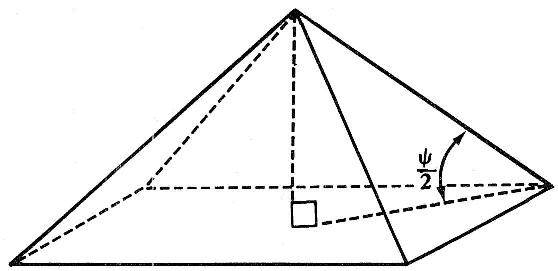


Fig. 2

Figure 2 shows one other place where the new series limit is found in the Great Pyramid. From the corner angles the outside edges of the Pyramid that follow the diagonals form the series limit divided by two as shown and expressed in radian measure.

To find any number in the new series, the recursion formula

$$F^*_{n+1} = F^*_n + F^*_{n-2} \tag{1}$$

can be used, where

$$F^*_0 = 0, \quad F^*_1 = F^*_2 = F^*_3 = 1.$$

The ratio, $\frac{F^*_{n+1}}{F^*_n}$, reaches a definite limit as one uses latter numbers of the series. This ratio is

$$\psi = \lim_{n\to\infty} \frac{F^*_{n+1}}{F^*_n} = 1.46557123\ldots\,. \tag{2}$$

Further investigation reveals that

$$\psi^3 - \psi^2 - 1 = 0 \tag{3}$$

and

$$\psi^{n+1} = \psi^2 F^*_n + \psi F^*_{n-2} + F^*_{n-1}. \tag{4}$$

Equation (3) reveals that ψ is a root of the equation

$$X^3 - X^2 - 1 = 0 \tag{5}$$

The roots of (5) are of considerable interest. These are easily verified to be the true roots of (5). Let the roots be α, β, and γ.

$$\alpha = \psi \tag{6}$$

$$\beta = -\frac{1}{2\psi^2}\left(1 + i\sqrt{4\psi^3 - 1}\right) \tag{7}$$

$$\gamma = -\frac{1}{2\psi^2}\left(1 - i\sqrt{4\psi^3 - 1}\right) \tag{8}$$

Roots (6), (7), and (8) will be used in (4) to develop a formula for F_m^*.

$$\alpha^{n+1} = \alpha^2 F_n^* + \alpha F_{n-2}^* + F_{n-1}^* \tag{9}$$

$$\beta^{n+1} = \beta^2 F_n^* + \beta F_{n-2}^* + F_{n-1}^* \tag{10}$$

$$\gamma^{n+1} = \gamma^2 F_n^* + \gamma F_{n-2}^* + F_{n-1}^* \tag{11}$$

Solving,

$$F_n^* = \frac{\begin{vmatrix} \alpha^{n+1} & \alpha & 1 \\ \beta^{n+1} & \beta & 1 \\ \gamma^{n+1} & \gamma & 1 \end{vmatrix}}{\begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}} \tag{12}$$

yielding,

$$F_n^* = \frac{\alpha^{n+1}(\beta - \gamma) + \beta^{n+1}(\gamma - \alpha) + \gamma^{n+1}(\alpha - \beta)}{\alpha^2(\beta - \gamma) + \beta^2(\gamma - \alpha) + \gamma^2(\alpha - \beta)} \tag{13}$$

which reduces to

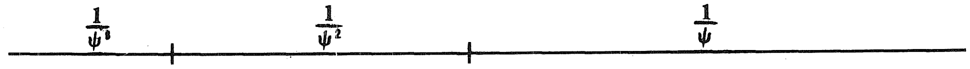
$$F_n^* = \frac{\alpha^{n+1}(\beta - \gamma) + \beta^{n+1}(\gamma - \alpha) + \gamma^{n+1}(\alpha - \beta)}{-i\sqrt{31}} \tag{14}$$

Equation (14) successfully computes F_n^*. The algebra gets fairly involved for higher numbers of the series, but the results agree with the established series.

Geometrical considerations are next. If one considers the relationship

$$\frac{1}{\psi^3} + \frac{1}{\psi^2} + \frac{1}{\psi} = \psi, \tag{15}$$

a line of length ψ can be thought of as divided into three parts as indicated on the left side of (15).



Line of Length ψ

Fig. 3

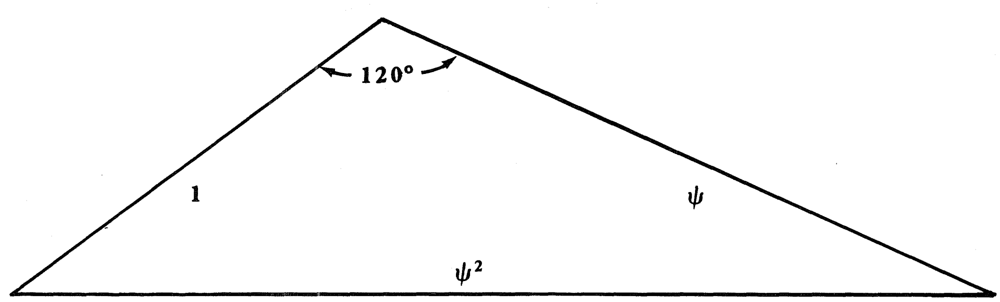
The result of (15) also leads to another interesting fact. The parts of the line in Figure 3 can be used to establish a proportion

$$\frac{1}{\psi^3}:\frac{1}{\psi^2}:\frac{1}{\psi} \tag{16}$$

which is better represented by the proportion

$$\psi^2:\psi:1 \tag{17}$$

The proportion in (17) established the sides of a special triangle which will be named the $\psi^2:\psi:1$ triangle.



$\psi^2:\psi:1$ Triangle

Fig. 4

The $\psi^2:\psi:1$ triangle incorporates an angle of 120° as its largest angle. This fact suggests that it can be placed into the vertices of the regular hexagon.

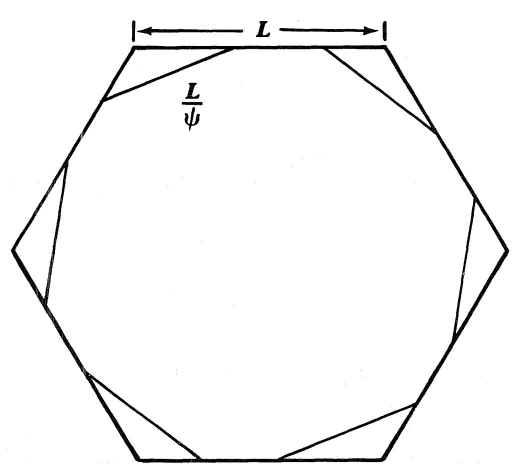


Fig. 5

Figure 5 represents a unit hexagon with six of the $\psi^2:\psi:1$ triangles at the vertices such that $\frac{L}{\psi}:\frac{L}{\psi^2}:\frac{L}{\psi^3}$ proportions are maintained in each.

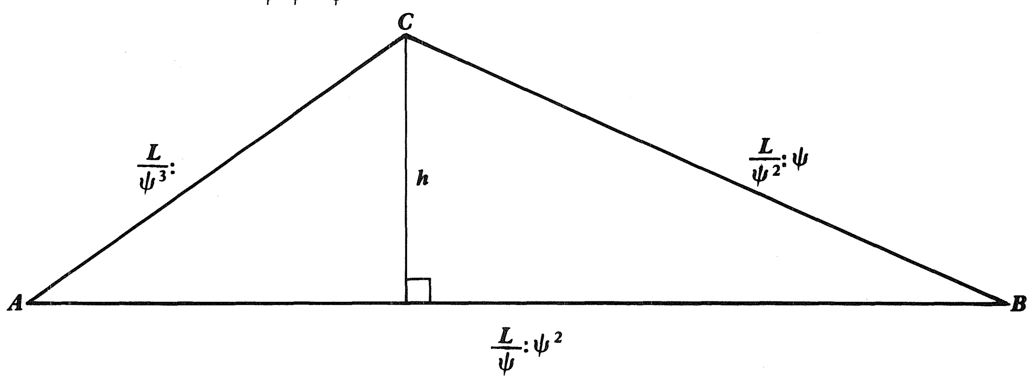


Fig. 6

Using Figure 6, we see that

$$\sin A = \frac{\sin C}{\psi} = \frac{\sqrt{3}}{2\psi}. \tag{18}$$

and

$$h = \frac{\sqrt{3}L}{2\psi^4} \tag{19}$$

The area of each small triangle in Figure 5 then is

$$\text{area} = \frac{1}{2}\,\frac{L}{\psi}\,\frac{\sqrt{3}L}{2\psi^4} = \frac{\sqrt{3}L^2}{4\psi^5}, \tag{20}$$

and for the total area represented by the six triangles

$$\text{area total} = \frac{3\sqrt{3}L^2}{2\psi^5}; \tag{21}$$

the area of the hexagon

$$\text{area hexagon} = \frac{3\sqrt{3}L^2}{2}. \tag{22}$$

A comparison of (22) to (21) yields

$$\frac{\text{area hexagon}}{\text{area six triangles}} = \psi^5 \tag{23}$$

This further indicates that the area of each small $\psi^2:\psi:1$ triangle is given by

$$\text{area of small } \psi^2:\psi:1 \text{ triangle} = \frac{\text{area hexagon}}{6\psi^5} \tag{24}$$

Rearranging equation (3),

$$\psi^3 = \psi^2 + 1 \tag{3}$$

gives the suggestion of volume as indicated in Figure 7.

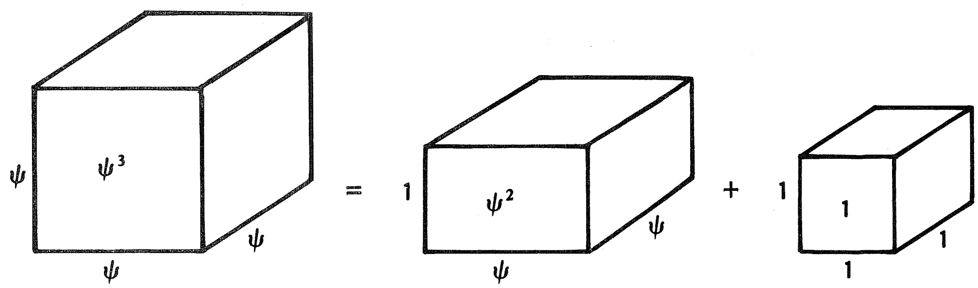


Fig. 7

The next property of the series limit is stated as a theorem.

Theorem: Given any triangle, choose any one of its sides and divide the length of that side by the factor ψ, the resulting length by ψ; and the final resulting length by ψ, so as to have three new lengths from the original side of the triangle. The three resulting lengths, when placed inside the triangle parallel to the side chosen will create equal perpendicular distances between the longest resulting length and side chosen as well as the shortest length of the vertex opposite the chosen side.

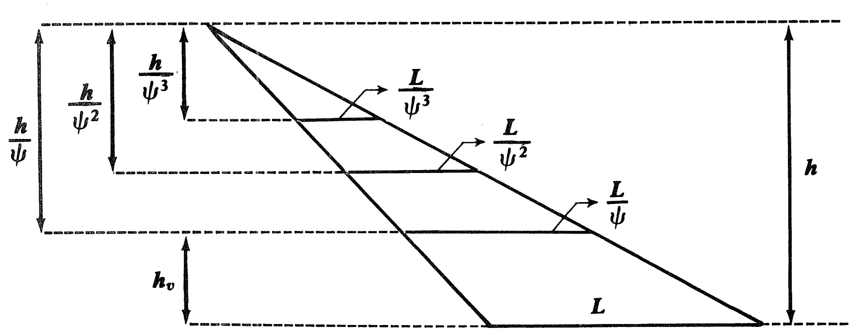


Fig. 8

Figure 8 represents one orientation of a given triangle with a side L. It is to be shown that

$$h_v = \frac{h}{\psi^3} \tag{25}$$

By similar triangles the height of the topmost triangle is $\frac{h}{\psi^3}$; the second is $\frac{h}{\psi^2}$; the third is $\frac{h}{\psi}$. It is easily seen that

$$h - \frac{h}{\psi} = h_v$$

or

$$h\left(1 - \frac{1}{\psi}\right) = h_v. \tag{26}$$

From the identity in (3),

$$\psi^3 - \psi^2 - 1 = 0, \tag{3}$$

then

$$1 = \psi^3 - \psi^2$$

and

$$\frac{1}{\psi^3} = 1 - \frac{1}{\psi}. \tag{27}$$

Substituting (27) into (26) yields

$$h_v = \frac{h}{\psi^3}. \tag{25}$$

A similar analysis can be used to prove the other orientations of the triangle.

Clarles Funk-Hellet [3], a French mathematician, constructed an additive series similar to Fibonacci's by replacing the second one in the series by a five and adding as in the original series. The series was developed into 36 rows, each row containing 18 entries. Table 1 illustrates Funk-Hellet's results in part.

Row 2 of the table contains the $1/\phi$, 1, ϕ, and ϕ^2 values, while the 14th entry of the 24th row yields a very precise value for π. The 7th column represents the one-eighth divisions of a circle. Other results were found by Funk-Hellet concerning other matters.

We construct a similar series using the series concerned in this paper by replacing the third one by six and adding as in the original series. Table 2 shows some of the results. This table was constructed in the same manner as Funk-Hellet's.

The 10th, 11th, and 12th entries of row 1 are values for $1/5\psi^3$, $1/5\psi^2$, and $1/5\psi$, respectively; the 14th through 18th places represent 2ψ, $2\psi^2$, $2\psi^3$, $2\psi^4$, and $2\psi^5$. The last entries of the 5th row give values for $1/\psi^5$, $1/\psi^4$, ..., $1/\psi$, 1, ψ, ..., ψ^4, ψ^5. The 9th entry of the 26th row represents $\phi - 1/2$; the 9th entry of the 29th row represents one-half the value of twice the height of the Great Pyramid less its base. The 6th entry of the 31st row yields the value for the log e.

One might wonder why Funk-Hellet chose to add the number five in his table and six was chosen in the newest case. It could be because the pentagon relates the Fibonacci limit and the hexagon relates the ψ-number limit. For whatever reason, the chosen numbers in conjunction with the related series to each yield some unexpected results.

As a final note:

$$\alpha_1 = \psi = \frac{1}{6}\left[4(29 + 3\sqrt{3}\sqrt{31})\right]^{1/3} + \frac{2}{3}\left[4(29 + 3\sqrt{3}\sqrt{31})\right]^{-1/3} + \frac{1}{3} \tag{6}$$

REFERENCES

1. Marjorie Bicknell, "A Primer for the Fibonacci Numbers: Part VIII. Sequences of Sums from Pascal's Triangle," *The Fibonacci Quarterly*, Vol. 9, No. 1 (February 1971), pp. 74-81.
2. H. S. M. Coxeter, *Introduction to Geometry* (New York: John Wiley & Sons, Inc., 1961), pp. 165-168.

Table 1

I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII
1	5	6	11	17	28	45	73	118	191	309	500	809	1309	2118	3427	5545	8972
2	10	12	22	34	56	90	146	236	382	618	1000	1618	2618	4236	6854	11090	17944
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
24	120	144	264	408	672	1080	1752	2832	4584	7416	12000	19416	31416	50832	82248	133080	215328
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Table 2

I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII
1	1	6	7	8	14	21	29	43	64	93	136	200	293	429	629	922	1351
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
5	5	30	35	40	70	105	145	215	320	465	680	1000	1465	2145	3145	4610	6755
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
26	26	156	182	208	364	546	754	1118	1664	2418	3536	5200	7618	11154	16354	23972	35126
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
29	29	174	203	232	406	609	881	1247	1856	2697	3944	5800	8497	12441	18241	26738	39179
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
31	31	186	217	248	434	651	899	1333	1984	2883	4216	6200	9083	13299	19499	28582	41881
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

3. Charles Funk-Hellet, *La Bible et la Grande Pyramide d'Egypte, temoignages authentiques du metre et de pi* (Montreal: Hellet Vincent, 1956).
4. Peter Tompkins, *Secrets of the Great Pyramid* (New York: Harper & Row, Publishers, 1971), pp. 189-200, 262-267.
5. A special thanks to the Hewlett-Packard Company for the development of their HP-35 calculator. This paper would not have been possible without its use.

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