ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS *to* Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. *This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within 2 months after publication of the problems.*

H-285 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA.*

Consider two sequences $\{H_n\}_{n=1}^{\infty}$ and $\{G_n\}_{n=1}^{\infty}$ such that

(a) $(H_n, H_{n+1}) = 1$,
(b) $(G_n, G_{n+1}) = 1$,
(c) $H_{n+2} = H_{n+1} + H_n \quad (n \geq 1)$, and
(d) $H_{n+1} + H_{n-1} = sG_n \quad (n \geq 1)$,
where s is independent of n.

Show $s = 1$ or $s = 5$.

H-286 *Proposed by P. Bruckman, Concord, CA.*

Prove the following congruences:

(1) $F_{5^n} \equiv 5^n \pmod{5^{n+3}}$;

(2) $F_{5^n} \equiv L_{5^{n+1}} \pmod{5^{2n+1}}$, $n = 0, 1, 2, \ldots$.

H-287 *Proposed by A. Mullin, Ft. Hood, Texas.*

Suppose $g(\cdot)$ is *any* strictly-positive, real-valued arithmetic function satisfying the functional equation:

$$\bigl(g(n + 1)/(n + 1)\bigr) + n = (n + 1)g(n)/g(n + 1)$$

for every integer n exceeding some prescribed positive integer m. Then $g(n)$ is *necessarily* asymptotic to $\pi(n)$, the number of prime numbers not exceeding n; i.e., $g(n) \sim \pi(n)$.

H-288 *Proposed by G. Wulczyn, Bucknell University, Lewisburg, PA.*

Establish the identities:

(a) $F_k L_{k+6r+3}^2 - F_{k+8r+4} L_{k+2r+1}^2 = (-1)^{k+1} L_{2r+1}^3 F_{2r+1} L_{k+4r+2}$.

(b) $F_k L_{k+6r}^2 - F_{k+8r} L_{k+2r}^2 = (-1)^{k+1} L_{2r}^3 F_{2r} L_{k+4r}$.

H-289 *Proposed by L. Carlitz, Duke University, Durham, N.C.*

Put the multinomial coefficient

$$(m_1, m_2, \ldots, m_k) = \frac{(m_1 + m_2 + \cdots + m_k)!}{m_1! m_2! \ldots m_k!}.$$

Show that

$$(*) \sum_{r+s+t=\lambda} (r,\ s,\ t)(m - 2r,\ n - 2s,\ p - 2t)$$

$$= \sum_{i+j+k+u=\lambda} (-2)^{i+j+k}(i, j, k, u)(m-j-k, n-k-i, p-i-j)(m+n+p \geq 2\lambda).$$

SOLUTIONS

A PAIR OF SUM SEQUENCES

H-269 *Proposed by G. Berzsenyi, Lamar University, Beaumont, Texas.*

The sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$, defined by

$$a_n = \sum_{k=0}^{[n/3]} \binom{n-2k}{k} \text{ and } b_{2n} = \sum_{k=0}^{[n/2]} \begin{bmatrix} n-2k \\ 2k \end{bmatrix},$$

$$b_{2n+1} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \begin{bmatrix} n-k \\ 2k+1 \end{bmatrix},$$

are obtained as diagonal sums from Pascal's triangle and from a similar triangular array of numbers formed by the coefficients of powers of x in the expansion of $(x^2 + x + 1)^n$, respectively.

$\left(\text{More precisely, } \begin{bmatrix} n \\ k \end{bmatrix} \text{ is the coefficient of } x^k \text{ in } (x^2 + x + 1)^n.\right)$
Verify that $a_n = b_{n-1} + b_n$ for each $n = 1, 2, \ldots$.

Solution by A. Shannon, School of Math Sciences, New South Wales Institute of Technology, Broadway, Australia.

It follows from Equation (4.1) of Shannon [2] with $P = R = 1$, $Q = 0$, that $a_n = a_{n+1} + a_{n+3}$.

A Pascal triangle for $\begin{bmatrix} n \\ k \end{bmatrix}$ can be set up as follows,

n \ k	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1	1								
2	1	2	3	2	1						
3	1	3	6	7	6	3	1				
4	1	4	10	16	19	16	10	4	1		
5	1	5	15	30	45	51	45	30	15	5	1

and it can be observed, and readily proved by induction that,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k-2 \end{bmatrix}.$$

By an extension of the methods of Carlitz [1] we can establish with somewhat tedious detail that

$$b_n = b_{n-2} + b_{n-3} + b_{n-4}.$$

Then, again with inductive methods, we get

$$a_n = a_{n-1} + a_{n-3} = b_{n-1} + b_{n-2} + b_{n-3} + b_{n-4} = b_{n-1} + b_n,$$

as required.

REFERENCES

1. L. Carlitz, "Some Multiple Sums and Binomial Identities," *S.I.A.M. J. Appl. Math.*, Vol. 13 (1965), pp. 469-486.
2. A. G. Shannon, "Iterative Formulas Associated with Generalized Third-Order Recurrence Relations," *S.I.A.M. J. Appl. Math.*, Vol. 23 (1972), pp. 364-368.

Also solved by P. Bruckman and the proposer.

IT'S A SINH
(Corrected)

H-270 *Proposed by L. Carlitz, Duke University, Durham, N.C.*

Sum the series

$$S \equiv \sum_{a,b,c} \frac{x^a y^b z^c}{(b+c-a)!(c+a-b)!(a+b-c)!}$$

where the summation is over all nonnegative a, b, c such that

$$a \le b + c,\ b \le a + c,\ c \le a + b.$$

Solution by P. Bruckman, Concord, CA.

Let $r = b + c - a$, $s = a + c - b$, and $t = a + b - c$. Then, $r + s = 2c$, $s + t = 2a$, $r + t = 2b$; this implies that r, s, and t are either all even or all odd. Hence,

$$(1)\quad S = \sum_{\substack{r,s,t \ge 0 \\ r \equiv s \equiv t \pmod 2}} \frac{x^{\frac{1}{2}(s+t)}}{r!} \frac{y^{\frac{1}{2}(t+r)}}{s!} \frac{z^{\frac{1}{2}(r+s)}}{t!}$$

Thus, $S = S_1 + S_2$, where

$$(2)\quad S_1 = \sum_{r,s,t \ge 0} \frac{x^{s+t}}{(2r)!} \frac{y^{t+r}}{(2s)!} \frac{z^{r+s}}{(2t)!},$$

$$(3)\quad S_2 = \sum_{r,s,t \ge 0} \frac{x^{s+t+1}}{(2r+1)!} \frac{y^{t+r+1}}{(2s+1)!} \frac{z^{r+s+1}}{(2t+1)!}$$

But S_1 and S_2 are readily evaluated, namely:

$$S_1 = \sum_{r,s,t \ge 0} \frac{(\sqrt{yz})^{2r}}{(2r)!} \frac{(\sqrt{xz})^{2s}}{(2s)!} \frac{(\sqrt{xy})^{2t}}{(2t)!} = \cosh\sqrt{yz} \cdot \cosh\sqrt{xz} \cdot \cosh\sqrt{xy},$$

and

$$S_2 = \sum_{r,s,t \geq 0} \frac{(\sqrt{yz})^{2r+1}}{(2r+1)!} \frac{(\sqrt{xz})^{2s+1}}{(2s+1)!} \frac{(\sqrt{xy})^{2t+1}}{(2t+1)!} = \sinh\sqrt{yz} \cdot \sinh\sqrt{xz} \cdot \sinh\sqrt{xy}.$$

Therefore,

$$(4)\quad S = \cosh\sqrt{xy} \cdot \cosh\sqrt{yz} \cdot \cosh\sqrt{zx} + \sinh\sqrt{xy} \cdot \sinh\sqrt{yz} \cdot \sinh\sqrt{zx}.$$

Also solved by W. Brady and the proposer.

H-271 (corrected)

Proposed by R. Whitney, Lock Haven State College, Lock Haven, PA.

Define the binary dual, D, as follows:

$$D = \left\{ t \,\middle|\, t = \prod_{i=0} (a_i + 2i);\ a_i \in \{0, 1\};\ n \geq 0 \right\}.$$

Let $\overline{D}$ denote the complement of D with respect to the set of positive integers. Form a sequence, $\{S_n\}_{n=1}^{\infty}$, by arranging $\overline{D}$ in increasing order. Find a formula for S_n.

(Note: The elements of D result from interchanging + and × in a binary number.)

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