ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS *to* Professor A. P. Hillman, 709 Solano Dr., S.E., Albuquerque, New Mexico 87108. *Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within 4 months of the publication date.*

DEFINITIONS

The Fibonacci numbers F and Lucas numbers L satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$. Also a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-382 *Proposed by A. G. Shannon, N.S.W. Institute of Technology, Australia.*

Prove that L has the same last digit (i.e., units digit) for all n in the infinite geometric progression

$$4,\ 8,\ 16,\ 32,\ \ldots .$$

B-383 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Solve the difference equation

$$U_{n+2} - 5U_{n+1} + 6U_n = F_n$$

B-384 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Establish the identity

$$F_{n+10}^4 = 55(F_{n+8}^4 - F_{n+2}^4) - 385(F_{n+6}^4 - F_{n+4}^4) + F_n^4.$$

B-385 *Proposed by Herta T. Freitag, Roanoke, VA.*

Let $T_n = n(n+1)/2$. For how many positive integers n does one have both $10^6 < T_n < 2 \cdot 10^6$ and $T_n \equiv 8 \pmod{10}$?

B-386 *Proposed by Lawrence Somer, Washington, D.C.*

Let p be a prime and let the least positive integer m with $F_m \equiv 0 \pmod{p}$ be an even integer $2k$. Prove that $F_{n+1}L_{n+k} \equiv F_n L_{n+k+1} \pmod{p}$. Generalize to other sequences, if possible.

B-387 *Proposed by George Berzsenyi, Lamar University, Beaumont, TX.*

Prove that there are infinitely many ordered triples of positive integers (x, y, z) such that

$$3x^2 - y^2 - z^2 = 1.$$

SOLUTIONS

ALMOST ALWAYS COMPOSITE

B-358 *Proposed by Phil Mana, Albuquerque, New Mexico.*

Prove that the integer u_n such that $u_n \leq n^2/3 < u + 1$ is a prime for only a finite number of positive integers n. (Note that $u_n = [n^2/3]$, where $[x]$ is the greatest integer in x and $u_1 = 0$, $u_2 = 1$, $u_3 = 3$, $u_4 = 5$, and $u_5 = 8$.)

Solution by Graham Lord, Université Laval, Québec.

If $n = 3m$, $3m + 1$, or $3m + 2$, where $m = 0, 1, 2, \ldots$, then, $u_n = 3m^2$, $m(3m + 2)$ or $(m + 1)(3m + 1)$, respectively. Thus, the only values of u_n that are prime are 3 and 5.

Also solved by George Berzsenyi, Paul S. Bruckman, Roger Engle & Sahib Singh, Herta T. Freitag, Bob Prielipp, and the proposer.

TRIBONACCI SEQUENCE

B-359 *Proposed by R. S. Field, Santa Monica, CA.*

Find the first three terms T_1, T_2, and T_3 of a Tribonacci sequence of positive integers $\{T_n\}$ for which

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n \quad \text{and} \quad \sum_{n=1}^{\infty} (T_n/10^n) = 1/T_4.$$

Solution by Graham Lord, Université Laval, Québec.

If $S(x) = \sum_{n=1}^{\infty} T_n x^n$, then

$$S(x) = [T_1(x - x^2 - x^3) + T_2(x^2 - x^3) + T_3 x\ ]/(1 - x - x^2 - x^3),$$

and, in particular,

$$S(1/10) = (89T_1 + 9T_2 + T_3)/889.$$

Hence,

$$T_4(89T_1 + 9T_2 + T_3) = 889 = 7 \cdot 127.$$

Since $T_4 = T_3 + T_2 + T_1 \geq 3$, it must be the smaller prime factor, 7, and

$$89T_1 + 9T_2 + T_3 = 127.$$

Thus, $T_1 = 1$, $T_2 = 4$, and $T_3 = 2$.

Also solved by George Berzsenyi, Michael Brozinski, Paul S. Bruckman, Roger Engle & Benjamin Freed & Sahib Singh, Charles B. Shields, and the proposer.

APPLYING QUATERNION NORMS

B-360 *Proposed by T. O'Callahan, Aerojet Manufacturing Co., Fullerton, CA.*

Show that for all integers a, b, c, d, e, f, g, h there exist integers w, x, y, z such that

$$(a^2 + 2b^2 + 3c^2 + 6d^2)(e^2 + 2f^2 + 3g^2 + 6h^2) = (w^2 + 2x^2 + 3y^2 + 6z^2).$$

Solution by Roger Engle & Sahib Singh, Clarion State College, Clarion, PA.

Defining the real quaternions A and B as

$$A = a + (\sqrt{2}b)i + (\sqrt{3}c)j + (\sqrt{6}d)k,$$

$$B = e + (\sqrt{2}f)i + (\sqrt{3}g)j + (\sqrt{6}h)k$$

and using the multiplicative property of norm N, namely $N(AB) = N(A)N(B)$, we conclude by comparison that

$$w = ae - 2bf - 3cg - 6dh, \quad x = af + be + 3ch - 3dg,$$

$$y = ag - 2bh + ce + 2df, \quad z = ah + bg - cf + de.$$

Also solved by Paul S. Bruckman, Bob Prielipp, Gregory Wulczyn, and the proposer.

A RATIONAL FUNCTION

B-361 *Proposed by L. Carlitz, Duke University, Durham, N.C.*

Show that

$$\sum_{r,s=0}^{\infty} x^r y^s u^{\min(r,s)} v^{\max(r,s)}$$

is a rational function of x, y, u, and v when these four variables are less than 1 in absolute value.

Solution by Roger Engle & Sahib Singh, Clarion State College, Clarion, PA.

If S denotes the required sum, then

$$S = \sum_{i=0}^{\infty}(xv)^i + \sum_{i=1}^{\infty}(yv)^i + xyuvS$$

$$\therefore S(1 - xyuv) = \frac{1}{1 - xv} + \frac{yv}{1 - yv}$$

$$\therefore S = \frac{1 - xyv^2}{(1 - xv)(1 - yv)(1 - xyuv)}$$

Also solved by Paul S. Bruckman, Robert M. Giuli, Graham Lord, and proposer.

TRIANGULAR NUMBER RESIDUES

B-362 *Proposed by Herta T. Freitag, Roanoke, VA.*

Let m be an integer greater than one (1) and let R_n be the remainder when the triangular number $T_n = n(n+1)/2$ is divided by m. Show that the sequence $R_0, R_1, R_2, \ldots$ repeats in a block $R_0, R_1, \ldots, R_t$ which reads the same from right to left as it does from left to right. (For example, if $m = 7$ then the smallest repeating block is 0, 1, 3, 6, 3, 1, 0.)

Solution by Graham Lord, Université Laval, Québec.

Since $T_{n+2m} = T_n + m(2n + 1 + 2m)$ then $R_n = R_{n+2m}$: the sequence repeats in blocks. And for $0 \le n < m$, as $T_{2m-n-1} = T_n + m(2m - 2n - 1)$ it follows that $R_n = R_{2m-n-1}$, which implies the reflecting property.

Note that if m is even the period is $2m$, since neither T_m nor T_2 is congruent to 0 modulo m. And if m is odd the period is m. The latter is proven thus: As $T_{n+m} \equiv T_n \pmod{m}$, the period, d, must divide m. But, by the reflecting property and the periodicity $T_0 \equiv T_{d-1} \equiv T_d \pmod{m}$, that is, m divides $T_d - T_{d-1} = d$. Hence, $d = m$.

Also solved by George Berzsenyi, Paul S. Bruckman, Roger Engle & Sahib Singh, Bob Prielipp, Gregory Wulczyn, and the propeser.

OVERLAPPING PALINDROMIC BLOCKS

B-363 *Proposed by Herta T. Freitag, Roanoke, VA.*

Do the sequences of squares $S_n = n^2$ and of pentagonal numbers $P_n = n(3n - 1)/2$ also have the symmetry property stated in B-362 for their residues modulo m?

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, PA.

For this symmetry property, it is necessary that two consecutive members of S_n or P_n be congruent to zero modulo m.

(a) $S_n = n^2$, $S_{n+1} = (n + 1)^2$.

Since $(n, n + 1) = 1$, S_n does not have the symmetry property of B-362.

(b) $P_n = \frac{n}{2}(3n - 1)$, $P_{n+1} = \frac{n + 1}{2}(3n + 2)$, $P_n = 1, 5, 12, 22, 35, \ldots$.

For any factor m of n, $(n, n + 1) = 1$, $(n, 3n + 2) = 1, 2$.
For any factor m of $3n - 1$, $(3n - 1, 3n + 2) = 1$, $(3n - 1, n + 1) = 1, 2, 4$.

Since the only common factor to P_n and P_{n+1} is 2, P_n does not have the symmetry property of B-362.

Also solved by Paul S. Bruckman, Roger Engle & Sahib Singh, Graham Lord, Bob Prielipp, and the proposer.

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