If $p_k \geq 73$, then, as in the last paragraph of the proof of (i), we have

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{5} + \frac{1}{5^2}\right) - \log\left(1 + \frac{1}{31} + \frac{1}{31^2} + \frac{1}{31^3} + \frac{1}{31^4}\right) + \frac{1}{5} + \frac{1}{31} + \frac{1}{2 \cdot 73^2} < b.$$

Finally, suppose $\alpha_1 \geq 4$. Then $p_k \geq 13$ and, as in the preceding paragraph,

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4}\right) + \frac{1}{5} + \frac{1}{2 \cdot 13^2} < b.$$

This completes the proof of (ii).

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REFERENCES

1. M. Buxton & S. Elmore, "An Extension of Lower Bounds for Odd Perfect Numbers," *Notices Amer. Math. Soc.*, Vol. 23 (1976), p. A-55.
2. D. B. Gillies, "Three New Mersenne Primes and a Statistical Theory," *Math. Comp.*, Vol. 18 (1964), pp. 93-97.
3. *Guiness Book of Records*, 22nd ed., 1975, p. 81.
4. P. Hagis, Jr., "Every Odd Perfect Number Has at Least Eight Prime Factors," *Notices Amer. Math. Soc.*, Vol. 22 (1975), p. A-60.
5. D. Suryanarayana, "On Odd Perfect Numbers II," *Proc. Amer. Math. Soc.*, Vol. 14 (1963), pp. 896-904.
6. D. Suryanarayana & P. Hagis, Jr., "A Theorem Concerning Odd Perfect Numbers," *The Fibonacci Quarterly*, Vol. 8, No. 3 (1970), pp. 337-346, 374.

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A SIMPLE CONTINUED FRACTION REPRESENTS A MEDIANT NEST OF INTERVALS

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1. While working on some mathematical aspects of the botanical problem of phyllotaxis, I came upon a property of simple continued fractions that is simple, pretty, useful, and easy to prove, but seems to have been overlooked in the literature. I present it here in the hope that it will be of interest to people who have occasion to teach continued fractions. The property is stated below as a theorem after some necessary terms are defined.

2. Terminology: For any positive integer n, let $n/0$ represent ∞. Let us designate as a "fraction" any positive rational number, or 0, or ∞, in the form a/b, where a and b are nonnegative integers, and either a or b is not zero. We say the fraction is in lowest terms if $(a, b) = 1$. Thus, 0 in lowest terms is $0/1$, and ∞ in lowest terms is $1/0$.

If inequality of fractions is defined in the usual way, that is

$$a/b < c/d \text{ if } ad < bc,$$

it follows that $x < \infty$ for $x = 0$ or any positive rational number.

3. *The Mediant*: If a/b and c/d are fractions in lowest terms, and $a/b < c/d$, the mediant between a/b and c/d is defined as $(a + c)/(b + d)$. Note that $a/b < (a + c)/(b + d) < c/d$.

Examples—The mediant between 1/2 and 1/3 is 2/5. If n is a nonnegative integer, the mediant between n and ∞ is $n + 1$. If n is a nonnegative integer and m is a positive integer, the mediant between n and $n+1/m$ is $n+1/(m+1)$.

4. *A Mediant Nest*: A mediant nest is a nest of closed intervals $I_0, I_1, \ldots, I_n, \ldots$ defined inductively as follows:

$$I_0 = [0, \infty].$$

For $n \geq 0$, if $I_n = [r, s]$, then I_{n+1} = either $[r, m]$ or $[m, s]$, where m is the mediant between r and s.

It is easily shown that if at least one I_n for $n \geq 1$ has for form $[r, m]$, then the length of I_n approaches 0 as $n \to \infty$, so that such a mediant nest is truly a nest of intervals, and it determines a unique number x that is contained in every interval of the nest. For the case where every I_n for $n \geq 1$ has the form $[m, s]$, let us say that the nest determines and "contains" the number ∞. Mediant nests are obviously related to Farey sequences.

5. *Long Notation for a Mediant Nest*: A mediant nest and the number it determines can be represented by a sequence of bits $b_1 b_2 b_3 \ldots b_i \ldots$, where, for $i > 0$, if $I_{i-1} = [r, s]$ and m is the mediant between r and s, $b_i = 0$ if $I_i = [r, m]$, and $b_i = 1$ if $I_i = [m, s]$.

Examples—$\dot{0} = 0$; $\dot{1} = \infty$; $\dot{1}\dot{0} = \tau$, the golden section; where each of these three examples is periodic, and the recurrent bits are indicated by the dots above them.

6. *Abbreviated Notation for a Mediant Nest*: The sequence of bits representing a mediant nest is a sequence of clusters of ones and zeros,

$$b_1 b_2 b_3 \ldots b_i \ldots = \overbrace{1 \ldots 1}^{a_1}\overbrace{0 \ldots 0}^{a_2}\overbrace{1 \ldots 1}^{a_3} \ldots$$

where the a_i indicate the number of bits in each cluster; $0 \leq a_1 \leq \infty$; $0 < a \leq \infty$ for $n > 1$; and the sequence (a_i) terminates with a_n if $a_n = \infty$. As an abbreviated notation for a mediant nest and the number x that it determines we shall write $x = (a_1, a_2, \ldots)$. Then $a_1 \leq x < a_1 + 1$. The sequence (a_i) terminates if and only if x is rational or ∞. Every positive rational number is represented by exactly two terminating sequences (a_i).

Examples—$(\infty) = \dot{1} = \infty$; $(0, \infty) = \dot{0} = 0$; $(0, 2, \infty) = 00\dot{1} = \frac{1}{2}$; $(0, 1, 1, \infty) = 01\dot{0} = \frac{1}{2}$. In general, if $x = (a_1, \ldots, a_{n-1}, a_n, \infty)$ where $a_n > 1$, then $x = (a_1, \ldots, a_{n-1}, a_n - 1, 1, \infty)$, and vice versa.

7. *Theorem*: If $x = (a_1, a_2, \ldots, a_n, \ldots)$, then $x = a_1 + 1/a_2 + \cdots + 1/a_n + \cdots$ and conversely. If $x = (a_1, \ldots, a_n, \infty)$, then $x = a_1 + 1/a_2 + \cdots + 1/a_n$ and conversely.

Proof of the Theorem:

I. The nonterminating case, $x = (a_1, a_2, \ldots, a_i, \ldots)$. Thus, x is irrational. Let p_i/q_i, for $i \geq 1$, be the principal convergents of $a_1 + 1/a_2 + \ldots$. Then a straightforward proof by induction establishes that for all even $i \geq 2$,

$$I_{a_1 + \cdots + a_i} = \left[p_{i-1}/q_{i-1},\ p_i/q_i\right],$$

and for all odd $i \geq 1$,

$$I_{a_1+\cdots+a_i} = \left[p_i/q_i,\ p_{i-1}/q_{i-1}\right].$$

Consequently, the nest determined by successive pairs of consecutive principal convergents of $a_1 + 1/a_2 + \cdots + 1/a_n + \cdots$ defines the same number as the mediant nest $(a_1, a_2, \ldots, a_n, \ldots)$.

II. The terminating case, $x = (a_1, \ldots, a_n, \infty)$. It follows from I that

$$I_{a_1+\cdots+a_{n+1}} = \left[p_n/q_n,\ p_{n+1}/q_{n+1}\right] \text{ or } \left[p_{n+1}/q_{n+1},\ p_n/q_n\right],$$

where

$$p_{n+1}/q_{n+1} = (p_{n-1} + a_{n+1}p_n)/(q_{n-1} + a_{n+1}q_n).$$

Since

$$\lim_{a_{n+1}\to\infty} p_{n+1}/q_{n+1} = p_n/q_n,$$

it follows that

$$x = \lim_{a_{n+1}\to\infty} I_{a_1+\cdots+a_{n+1}} = p_n/q_n = a_1 + 1/a_2 + \cdots + 1/a_n.$$

III. The "conversely" in the theorem follows from the fact that the mapping of the set of mediant nests into the set of simple continued fractions established in I and II is one-to-one and onto.

Example—The mediant nest $(0, 2, 3, \infty)$ and the continued fraction $0 + 1/2 + 1/3$ represent the same number. Verification:

a. $(0, 2, 3, \infty)$ is the abbreviated notation for the sequence of bits

$$00111\dot{0}.$$

The intervals I_n defined by this sequence of bits are:

Bit	*Interval*	*Mediant between Endpoints of Interval*
	$I_0 = [0/1,\ 1/0]$	$(0 + 1)/(1 + 0) = 1/1$
0	$I_1 = [0/1,\ 1/1]$	$(0 + 1)/(1 + 1) = 1/2$
0	$I_2 = [0/1,\ 1/2]$	$(0 + 1)/(1 + 2) = 1/3$
1	$I_3 = [1/3,\ 1/2]$	$(1 + 1)/(3 + 2) = 2/5$
1	$I_4 = [2/5,\ 1/2]$	$(2 + 1)/(5 + 2) = 3/7$
1	$I_5 = [3/7,\ 1/2]$	$(3 + 1)/(7 + 2) = 4/9$
0	$I_6 = [3/7,\ 4/9]$	$(3 + 4)/(7 + 9) = 7/16$
	$\vdots$	
0	$I_n = [3/7,\ m_{n-1}]$	$n \geq 6$, m_{n-1} = the mediant between the endpoints of I_{n-1}.

Since $\lim_{n\to\infty} m_{n-1} = 3/7$, the number defined by this mediant nest is 3/7.

b. The continued fraction

$$0 + \cfrac{1}{2 + \cfrac{1}{3}} = 3/7.$$

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