that $e < e_i$ and $V(p^e) \leq e - i$, where $i \in \{2, 3, 4\}$? (By the computational results in [5] we may conclude that this does not happen when $p < 30{,}000$.)

2. Is there a power $n = p^e$ of some irregular prime such that

$$V(p^e) \leq e - 5?$$

Final Remark: Professor L. Carlitz and Jack Levine in [3] asked similar questions about Euler numbers and polynomials. Analogous results about the periodicity of the sequence of the Bernoulli polynomials reduced modulo n and the polynomial functions over Z generated by the Bernoulli polynomials will be derived in a later paper.

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THE RANK-VECTOR OF A PARTITION

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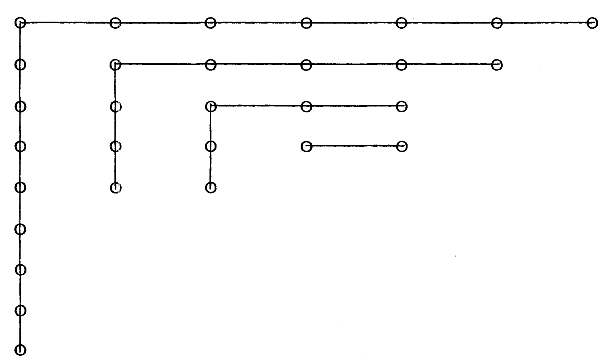
1. INTRODUCTION

The Ferrars graph of a partition may be regarded as a set of nested right angles of nodes. The depth of a graph is the number of right angles it has. For example, the graph



is four deep or is of depth four. It is clear that a graph of depth k cannot have less than k^2 nodes.

Denote by x_i the number of nodes on the horizontal, and by y_i the number of those on the vertical section of the ith right angle, starting with the outermost right angle as the first. Then, the partition can be very conveniently represented by the $2 \times k$ matrix:

$$\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_k \\ y_1 & y_2 & y_3 & \cdots & y_k \end{bmatrix}$$

or simply by

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix}_k .$$

Evidently, we must have

(1.1) $$x_i \geq x_{i+1} + 1, \quad y_i \geq y_{i+1} + 1, \quad i \leq k - 1.$$

It must be remembered that x's and y's are positive integers. The Atkin-ranks of the graph [1] are given by

(1.2) $$R_k = [x_1 - y_1, x_2 - y_2, \ldots, x_k - y_k] = [x_i - y_i]_k,$$

which we shall call the rank-vector both of the graph and of the partition it represents.

The number of nodes in the graph is given by

(1.3) $$n = \sum_{i=1}^{k} (x_i + y_i - 1).$$

In our graph, the matrix

$$\begin{bmatrix} 7 & 5 & 3 & 2 \\ 9 & 4 & 3 & 1 \end{bmatrix}$$

represents a partition of 30 and its rank-vector is

$$[-2 \quad 1 \quad 0 \quad 1].$$

Obviously, if R_k is the rank-vector of a partition, then the rank-vector of its conjugate partition is $-R_k$. Hence, the rank-vector of a self-conjugate partition of depth k must be $[0]_k$.

Again, if $[r_i]_k$ is the rank-vector of the partition given by $\begin{bmatrix} x_i \\ y_i \end{bmatrix}_k$, then we have

(1.4) $$y_i = x_i - r_i .$$

2. SOME CONSEQUENCES OF (1.1)

Since $y_i \geq y_{i+1} + 1$, we must have $x_i - r_i \geq x_{i+1} - r_{i+1} + 1$. Hence, for each $i \leq k - 1$,

(2.1) $$x_i \geq \max(x_{i+1} + 1, x_{i+1} + r_i - r_{i+1} + 1).$$

Since y_k is a positive integer, we conclude that

(2.2) $$x_k \geq \max(r_k + 1, 1).$$

From (1.3) and (1.4), we further have

(2.3) $$\sum_{i=1}^{k} x_i = \frac{1}{2}\left(n + k + \sum_{i=1}^{k} r_i\right).$$

Hence a partition of n with a given rank-vector $[r_i]_k$ can exist only if n has the same parity as

$$k + \sum_{i=1}^{k} r_i .$$

In what follows, we assume that our n's satisfy this condition. Moreover, i shall invariably run over the integers from 1 to k.

3. THE BASIS OF A GIVEN RANK-VECTOR

There are an infinite number of Ferrars graphs which have the same rank-vector. All such graphs have the same depth but not the same number of nodes necessarily.

Theorem: Among the graphs with the same rank-vector, there is just one with the least number of nodes.

Proof: Using the equality sign in place of the sign $\geq$ in (2.2) and (2.1), we obtain the least value of each of the x_i's, $i \leq k$. (1.3) and (1.4) then give n_0 that is the least n for which a graph with the given rank-vector exists. This proves the theorem.

Incidentally, we also get the unique partition with the given rank-vector and the least number of nodes. We call this unique partition the basis of the given rank-vector.

Example: Let us find the basis of the rank-vector [-2 3 0 1]. With the equality sign in place of the of the inequality sign, (2.2) gives $x_4 = 2$. With the equality sign in place of $\geq$, (2.1) now gives, in succession,

$$x_3 = 3,\ x_2 = 7, \text{ and } x_1 = 8.$$

From (4) of Section 1, we now have

$$y_4 = 1,\ y_3 = 3,\ y_2 = 4, \text{ and } y_1 = 10.$$

Hence, the required basis is

$$\begin{bmatrix} 8 & 7 & 3 & 2 \\ 10 & 4 & 3 & 1 \end{bmatrix}.$$

This represents a partition of 34.

We leave the reader to verify the following two trivial-looking but very useful observations:

(a) If $\begin{bmatrix} x_i \\ y_i \end{bmatrix}$ is the basis of $[r_i]$ and h is an integer, then the basis of the vector $[r_i + h]$ is given by

$$\begin{bmatrix} x_i + h \\ y_i \end{bmatrix} \text{ or } \begin{bmatrix} x_i \\ y_i - h \end{bmatrix}$$

according as h is positive or negative.

(b) If $h_1 \geq h_2 \geq \dots \geq h_k \geq 0$ are integers, then the graphs of

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} \text{ and } \begin{bmatrix} x_i + h_i \\ y_i + h_i \end{bmatrix}$$

have the same rank-vector.

4. PARTITIONS OF n WITH A GIVEN RANK-VECTOR

Let $\begin{bmatrix} x_i \\ y_i \end{bmatrix}_k$ be the basis of the given rank-vector and n_0 the number of nodes in the basis. For our n to have any partitions with the given rank-vector, it is necessary that n has the same parity as n_0 and $n \geq n_0$. Assume that this is so. Write

$$m = \frac{1}{2}(n - n_0).$$

List all the partitions of m into at most k parts. Let

$$m = h_1 + h_2 + \cdots + h_k,$$

with $h_1 \geq h_2 \geq \ldots \geq h_k \geq 0$, be any such partition of m. Then the matrix

$$\begin{bmatrix} x_i + h_i \\ y_i + h_i \end{bmatrix} \tag{4.1}$$

provides a partition of n with the given rank-vector.

The one-one correspondence between the partitions of m and the matrices (4.1) establishes the following

Theorem: The number of partitions of n with the given rank-vector is the same as the number of partitions of m into at most k parts where m is as defined above.

Example: Let the given rank-vector be [-3 2 1 -1] and $n = 43$. Then the basis of the vector is readily seen to be

$$\begin{bmatrix} 7 & 6 & 4 & 1 \\ 10 & 4 & 3 & 2 \end{bmatrix}$$

so that $n_0 = 33$ and $m = 5$.

The partitions of 5 into at most 4 parts are:

5; 4 + 1, 3 + 2; 3 + 1 + 1, 2 + 2 + 1; 2 + 1 + 1 + 1.

Therefore, the required partitions of 43 are provided by the matrices:

$$\begin{bmatrix} 12 & 6 & 4 & 1 \\ 15 & 4 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 11 & 7 & 4 & 1 \\ 14 & 5 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 10 & 8 & 4 & 1 \\ 13 & 6 & 3 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} 10 & 7 & 5 & 1 \\ 13 & 5 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 9 & 8 & 5 & 1 \\ 12 & 6 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 9 & 7 & 5 & 2 \\ 12 & 5 & 4 & 3 \end{bmatrix}.$$

We leave it to the reader to see how the graphs of partitions of n can be constructed directly from that of the basis. As an exercise, he/she might also find a formula for the number of self-conjugate partitions of n.

As a corollary to the theorem of this section, we have

Corollary: The number of partitions of $n + hk$, $h > 0$, with rank-vector $[r_i + h]$ is the same as the number of partitions of n with rank-vector $[r_i]$.

This follows immediately from observation (a) in the preceding section.

5. THE BOUNDS FOR THE ATKIN-RANKS

What can be said concerning the Atkin-ranks of partitions of n for which $x_1 \leq a$, $y_1 \leq b$?

We show that these ranks are bounded both above and below. Since $x_1 \le a$, the number of rows a partition of n can occupy is not less than u, where

$$u - 1 < n/a \le u.$$

Hence, none of the ranks can exceed $(a - u)$.

Similarly, none of the ranks can fall short of $(v - b)$, where

$$v - 1 < n/b \le v.$$

Of course, for n to have a partition of said type, it is necessary to have

$$n \le ab.$$

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THE ANDREWS FORMULA FOR FIBONACCI NUMBERS

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1. In what follows: small letters denote integers; $n > 0$; p denotes an odd prime other than 5; [ ] is the greatest integer function; and for convenience, we write

$$(n;r) \text{ for } \binom{n}{r}.$$

The two relations

$$(n;r) = (n;n - r), \text{ and} \tag{1.1}$$

$$(n;r - 1) + (n;r) = (n + 1;r) \tag{1.2}$$

are freely used, and we take, as usual,

$$(t;0) = 1 \text{ for all integers } t, \text{ and}$$

$$(n;r) = 0 \text{ if } r > n, \text{ and also when } r \text{ is negative.}$$

We further define

$$S(n,r) = \sum_j (n;j), \tag{1.3}$$

where j runs over all nonnegative integers which are $\equiv r \pmod 5$.

As a consequence of this definition and the relations (1.1) and (1.2) we have

$$S(n,r) = S(n,n - r), \text{ and} \tag{1.4}$$

$$S(n,r - 1) + S(n,r) = S(n + 1,r). \tag{1.5}$$

2. The Fibonacci numbers F_n are defined by the relations

$$F_1 = 1 = F_2, \text{ and} \tag{2.1}$$

$$F_n + F_{n+1} = F_{n+2} \text{ for each } n \ge 1. \tag{2.2}$$