NEARLY LINEAR FUNCTIONS

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Let $\alpha = (1 + \sqrt{5})/2$, $[x]$ be the greatest integer in x, $a_1(n) = [\alpha n]$, and $a_2(n) = [\alpha^2 n]$. A partial table follows:

n	1	2	3	4	5	6	7	8	9	10	11
$a_1(n)$	1	3	4	6	8	9	11	12	14	16	17
$a_2(n)$	2	5	7	10	13	15	18	20	23	26	28

It is known (see [1]) that $a_1(n)$ and $a_2(n)$ form the nth safe-pair of Wythoff's variation on the game Nim. These sequences have many interesting properties and are closely connected with the Fibonacci numbers. For example, let

$$\sigma(n) = a_1(n + 1) - 1;$$

then

$$\sigma^2(n) = \sigma[\sigma(n)] = a_2(n + 1) - 2,$$

$$\sigma(F_n) = F_{n+1} \text{ for } n > 1,$$

and

$$\sigma(L_n) = L_{n+1} \text{ for } n > 2.$$

Here we generalize by letting d be in $\{2, 3, 4, \ldots\}$ and letting h_n be the dth-order generalized Fibonacci number defined by the initial conditions

$$\text{(I)} \qquad h_i = 2^{i-1} \quad \text{for} \quad 1 \le i \le d$$

and the recursion

$$\text{(R)} \qquad h_{n+d} = h_n + h_{n+1} + \cdots + h_{n+d-1}.$$

The recursion (R) easily implies

$$\text{(R}'\text{)} \qquad h_{n+d+1} = 2h_{n+d} - h_n \quad \text{or} \quad h_n = 2h_{n+d} - h_{n+d+1}.$$

The first of these is convenient for calculation of h_n for increasing values of n and the second for decreasing n.

Representations for integers as sums of distinct terms h_n will be used below to study some nearly linear functions from $N = \{0, 1, 2, \ldots\}$ to itself; these will include generalizations of the Wythoff sequences. Associated partitions of $Z^+ = \{1, 2, 3, \ldots\}$ will also be presented.

1. CHARACTERISTIC SEQUENCES

Let T be the set of all sequences $\{e_n\} = e_1, e_2, \ldots$ with each e_n in $\{0, 1\}$ and with an n_0 such that $e_n = 0$ for $n > n_0$. Let $z = z(E)$ be the smallest n with $e_n = 0$ and let E^* be the $\{e_n^*\}$ in T given by $e_n^* = 0$ for $n < z$, $e_z^* = 1$, and $e_n^* = e_n$ for $n > z$. If some $e_n = 1$, let $u(E)$ be the smallest such n.

If $E = \{e_n\}$ is in T and $Y = \{y_n\} = y_1, y_2, \ldots$ is any sequence of integers, then $e_1 y_1 + e_2 y_2 + \cdots$ is really a finite sum which we denote by $E \cdot Y$. For each integer j, let $H_j = \{h_{n+j}\} = h_{j+1}, h_{j+2}, \ldots$ where the h_n are defined by (I) and (R). Also, let $H = H_0$.

Lemma 1: Let $z = z(E)$ and $b = E^* \cdot H_j - E \cdot H_j$. Then

(a) $u(E^*) = z$.

(b) If $z = 1$, $b = h_{j+1}$. If $z > 1$, $b = h_{z+j} - h_{z+j-1} - h_{z+j-2} - \cdots - h_{j+1}$.

(c) If $1 \le z \le d$ and $j = 0$, $b = 1$.

Proof: Parts (a) and (b) follow immediately from the relevant definitions. Then (c) follows from (b), the initial conditions (I), and the fact that

$$1 + 2 + \cdots + 2^{z-2} = 2^{z-1} - 1.$$

2. THE SUBSET S OF T

Let S consist of the $\{c_n\}$ in T with

$$c_n c_{n+1} \cdots c_{n+d-1} = 0 \quad \text{for all } n \text{ in } Z^+.$$

Lemma 2: If C is in S then:

(a) $1 \le z(C) \le d$,
and
(b) $C^* \cdot H - C \cdot H = 1$.

Proof: Part (a) follows from the defining condition, with $n = 1$, for the subset S. Then Lemma 1(c) implies the present part (b).

Lemma 3: If $C \cdot H = C' \cdot H$ with C and C' in S, then $C = C'$.

Proof: Let $C = \{c_n\}$ and $C' = \{c_n'\}$. We assume $C \ne C'$ and seek a contradiction. Then $c_k \ne c_k'$ for some k, and there is a largest such k since $c_n = 0 = c_n'$ for n large enough. We use this maximal k and without loss of generality assume that $c_k = 0$ and $c_k' = 1$. Then

$$C' \cdot H - C \cdot H = \sum_{i=1}^{k} (c_i' - c_i)h_i \le h_k - \sum_{i=1}^{k-1} c_i h_i, \tag{1}$$

since $h_i > 0$ for $i > 0$. Let $k = qd + r$, where q and r are integers with $0 \le r < d$. Then one can use (R) to show that

$$h_k = (h_1 + h_2 + h_3 + \cdots + h_{k-1}) - (h_r + h_{r+d} + h_{r+2d} + \cdots + h_{k-d}) + 1. \tag{2}$$

(The interpretation of this formula when $1 \le k < d$ is not difficult.) Since $c_n = 0$ for at least one of any d consecutive values of n and $h_n < h_{n+1}$ for $n > 0$, (2) implies that

$$h_k > c_1h_1 + c_2h_2 + \cdots + c_{k-1}h_{k-1}.$$

This and (1) give us the contradiction $C' \cdot H > C \cdot H$. Hence $C' = C$, as desired.

Lemma 4: For every E in T there is a C in S such that:

(a) $E \cdot H_j = C \cdot H_j$ for all j,

(b) $z(E) \equiv z(C) \pmod{d}$,

(c) $u(E) \equiv u(C) \pmod{d}$.

(d) This C is uniquely determined by E.

Proof: We may assume that $E = \{e_n\}$ is not in S. Then

$$e_k e_{k+1} \cdots e_{k+d-1} = 1 \text{ for some } k.$$

There is a largest such k since $e_n = 0$ for large enough n. Using this maximal k, one has $e_{k+d} = 0$ and we let $E' = \{e'_n\}$ be given by $e'_n = 0$ for $k \leq n < k + d$, $e'_{k+d} = 1$, and $e'_n = e_n$ for all other n. The recursion (R) implies that $E \cdot H_j = E' \cdot H_j$ for all j. It is also clear that $z(E) \equiv z(E') \pmod{d}$ and $u(E) \equiv u(E') \pmod{d}$. If E' is not in S, we give it the same treatment given E. After a finite number of such steps, one obtains a C in S with the desired properties. Lemma 3 tells us that this C is uniquely determined by E.

3. THE BIJECTION BETWEEN N AND S

We next establish a 1-to-1 correspondence $m \longleftrightarrow C_m = \{c_{mn}\}$ between the nonnegative integers m and the sequences of S.

Lemma 5: S is a sequence $C_0, C_1, \ldots$ of sequences C_m such that $C_m \cdot H = m$ and $u(C_{m+1}) \equiv z(C_m) \pmod{d}$.

Proof: The only C in S with $C \cdot H = 0$ is

$$C_0 = \{c_{0n}\} = 0, 0, 0, \ldots .$$

Now, assume inductively that for some k in N there is a unique C_k in S with $C_k \cdot H = k$. Then Lemma 2(b) tells us that $C_k^* \cdot H = C_k \cdot H + 1 = k + 1$. It follows from Lemma 4 that there is a unique C_{k+1} in S with $C_{k+1} \cdot H = C_k^* \cdot H = k + 1$. Finally, $u(C_{m+1}) \equiv z(C_m) \pmod{d}$ is a consequence of Lemma 1(a) and Lemma 4(c). The desired results then follow by induction.

Lemma 6: Let E be in T and $E \cdot H = m$. Then $E \cdot H_j = C_m \cdot H_j$, for all j, $z(E) \equiv z(C_m) \pmod{d}$, and $u(E) \equiv u(C_m) \pmod{d}$.

Proof: Lemma 4 tells us that there us a C in S with $E \cdot H_j = C \cdot H_j$ for all integers j, $z(E) \equiv z(C) \pmod{d}$, and $u(E) \equiv u(C) \pmod{d}$. The hypothesis $E \cdot H = m$ and Lemma 5 then imply that $C = C_m$.

4. THE SHIFT FUNCTIONS

Let functions $\sigma^i(m)$ from $N = \{0, 1, \ldots\}$ into $Z = \{\ldots, -2, -1, 0, 1, \ldots\}$ be given for all integers i by

(3) $$\sigma^i(m) = C_m \cdot H_i.$$

That is, $\sigma^i(C_m \cdot H) = C_m \cdot H_i$. Using this, one sees easily that

$$\sigma^i[\sigma^j(m)] = \sigma^{i+j}(m)$$

for all integers i and j and all m in N. We also note that

$$\sigma^0(m) = C_m \cdot H = m.$$

Lemma 7:

(a) $\sigma^j(0) = 0$ and $\sigma^j(h_n) = h_{n+j}$ for all integers j and n.

(b) $\sigma^j(E \cdot H) = E \cdot H_j$ for all integers j and all E in T.

(c) If E and E' are in T, $E \cdot E' = 0$, $E \cdot H = m$, and $E' \cdot H = n$, then

$$\sigma^j(m + n) = \sigma^j(m) + \sigma^j(n) \text{ for all } j \text{ in } Z.$$

Proof: Part (a) is clear. Part (b) follows from (3) and Lemma 6. For (c), let $E = \{e_n\}$, $E' = \{e'_n\}$, and $y_n = e_n + e'_n$. The hypothesis $E \cdot E'$ implies that $Y = \{y_n\}$ is in T. Then $Y \cdot H = E \cdot H + E' \cdot H = m + n$. This and (b) tell us that $\sigma^j(m + n) = Y \cdot H_j$, which equals $E \cdot H_j + E' \cdot H_j = \sigma^j(m) + \sigma^j(n)$, as desired.

5. A PARTITION OF Z^+

For $i = 1, 2, \ldots, d$ let A_i be the set of all positive integers m for which $u(C_m) \equiv i \pmod d$. Clearly these A_i partition Z^+, i.e., they are disjoint and their union is Z^+.

Lemma 8: Let k be in A_i. Then $k = h_i + C \cdot H_i$ for some C in S.

Proof: Let $u(C_k) = u$. Then

(4) $$k = h_u + c_{k,u+1}h_{u+1} + \cdots = h_u + C' \cdot H_u \text{ for some } C' \text{ in } S.$$

Since k is in A_i, $u \equiv i \pmod d$. If $u > i$, we use (4) and the recursion (R) to obtain

$$k = h_{u-d} + h_{u-d+1} + \cdots + h_{u-1} + C' \cdot H_u = h_{u-d} + C'' \cdot H_{u-d},$$

with C'' in S.

If $u - d > i$, we continue this process until we have $k = h_i + C \cdot H_i$ with C in S. This completes the proof.

Now, for every integer j, we define a function a_j from Z^+ into Z by

$$a_j(n) = h_j + \sigma^j(n-1).$$

Clearly this means that, for m in N,

(5) $$a_j(m+1) = h_j + C_m \cdot H_j = h_j + c_{m1}h_{j+1} + c_{m2}h_{j+2} + \cdots.$$

It follows from (5) that, for constant k, $a_n(k)$ has the same recursion formulas as the h_n. In particular,

(6) $$a_{j+1}(n) = 2a_j(n) - a_{j-d}(n).$$

Lemma 9: $\{a_i(r) \mid r \in Z^+\} = A_i$ for $1 \le i \le d$.

Proof: Let r be in Z^+ and $m = r - 1$. One sees from (5) that

$$a = a_i(r) = a_i(m+1)$$

if of the form $E \cdot H$ with $u(E) = i$. Then $i \equiv u(C_a) \pmod d$ by Lemma 6. Hence a is in A_i.

Now let $k \in A_i$. Then Lemma 8 tells us that $k = h_i + C \cdot H_i$ with C in S. Let $C \cdot H = m$. Then $C = C_m$ and it follows from (5) that

$$k = a_i(m+1) \in \{a_i(r) \mid r \in Z^+\}.$$

This completes the proof.

6. SELF-GENERATING SEQUENCES

Next we define b_{ij} for $1 \le i \le d$ and all integers j by

(7) $$b_{1j} = h_{j+1},\ b_{ij} = h_{i+j} - h_{i+j-1} - h_{i+j-2} - \cdots - h_{j+1} \text{ for } 2 \le i \le d.$$

We will use these b_{ij} to show that the sets A_i are self-generating and to count the integers in $A_i \cap \{1, 2, \ldots, n\}$.

One can show that the b_{ij} could be defined alternatively by the initial conditions $b_{i0} = 1$ for $1 \le i \le d$ and the recursion formulas

$$b_{i,j+1} = b_{1j} + b_{i+1,j} \text{ for } 1 \le i < d;\ b_{d,j+1} = b_{1j} = b_{j+1}.$$

These show, for example, that

(8) $$b_{i1} = 2 \text{ for } 1 \le i < d \text{ and } b_{d1} = 1.$$

The definition (7) for b_{in} in terms of the h's implies that, for fixed i, the b_{in} satisfy the same recursion formulas as the h_n; in particular, one has

$$b_{in} = 2b_{i,n+d} - b_{i,n+d+1}.$$

This can be used to show that

$$b_{i,-i} = 1 \text{ for } 1 \le i \le d,\ b_{ij} = 0 \text{ for } -d \le j < 0 \text{ and } i \ne -j. \tag{9}$$

Theorem 1: Let $b_j(m) = a_j(m+1) - a_j(m)$. Then $b_j(m) = b_{ij}$ for m in A_i.

Proof: It follows from (5) that $b_j(m) = C_m \cdot H_j - C_{m-1} \cdot H_j$. In the proof of Lemma 5, we saw that $C_m \cdot H_j = C^*_{m-1} \cdot H_j$; hence

$$b_j(m) = C_{m-1} \cdot H_j - C^*_{m-1} \cdot H. \tag{10}$$

Let $u = u(C_m)$ and $z = z(C_{m-1})$. The hypothesis $m \in A_i$ means that $u \equiv i \pmod d$. Then $z \equiv i \pmod d$ by Lemma 5. This, the fact that $1 \le i \le d$, and Lemma 2(a) imply that $z = i$. Finally, $z = i$ and Lemma 1 tell us that the $b_j(m)$ of (10) is equal to the b_{ij} defined in (7).

Theorem 2: For $1 \le i \le d$, $b_{-i}(m)$ equals 1 when m is in A_i and equals 0 when m is not in A_i.

Proof: This follows from Theorem 1 and the formulas in (9).

Theorem 3: The number of integers in the intersection of A_i and $\{1, 2, \ldots, m\}$ is $a_{-i}(m+1)$ for $1 \le i < d$ and is $a_{-d}(m+1) - 1$ for $i = d$.

Proof: One sees that $a_{-i}(1) = h_{-i} + C_0 \cdot H_{-i} = h_{-i} = 0$ for $1 \le i < d$ and that $a_{-d}(1) = h_{-d} = 1$. It is also clear that

$$a_{-i}(m+1) = a_{-i}(1) + b_{-i}(1) + b_{-i}(2) + \cdots + b_{-i}(m).$$

This and Theorem 2 give us the desired result.

7. COMPOSITES

First we note that

$$a_i[a_j(n)] = h_i + \sigma^i[a_j(n) - 1] = h_i + \sigma^i[h_j - 1 + \sigma^j(n-1)]. \tag{11}$$

For $1 \le j \le d$, we have $h_j = 2^{j-1}$ and hence we have

$$h_j - 1 = h_1 + h_2 + \cdots + h_{j-1} \quad \text{for } 1 < j \le d.$$

Also, we know that $\sigma^j(n-1)$ is of form $c_1h_{j+1} + c_2h_{j+2} + \cdots$ with c_k in $\{0, 1\}$. Hence (11) leads to

$$\begin{aligned} a_i[a_j(n)] &= h_i + \sigma^i[h_1 + h_2 + \cdots + h_{j-1} + c_1h_{j+1} + \cdots] \\ &= h_i + h_{i+1} + h_{i+2} + \cdots + h_{i+j-1} + c_1h_{i+j+1} + \cdots \\ &= h_i + h_{i+1} + \cdots + h_{i+j-1} + \sigma^{i+j}(n-1) \\ &= h_i + h_{i+1} + \cdots + h_{i+j-1} - h_{i+j} + a_{i+j}(n) \end{aligned} \tag{12}$$

for $1 < j \le d$ and all integers i.

Letting $i = -d$ and using the facts that $h_{-d} = 1 = h_0$ and $h_n = 0$ for $-d < n < 0$, (12) implies that

$$a_{-d}[a_j(n)] = 1 + a_{j-d}(n) \quad \text{for } 1 \le j < d, \quad a_{-d}[a_d(n)] = a_0(n) = n. \tag{13}$$

Our derivation applies for $1 < j \le d$, but the result in (13) for $j = 1$ can also be seen to be true.

One may note that (12) implies

$$a_i[a_j(n)] - a_j[a_i(n)] = h_i + h_{i+1} + \cdots + h_{j-1} \text{ for } 1 \le i < j \le d.$$

Theorem 4: For $1 \le j < d$, $a_{j+1}(n)$ is $2a_j(n)$ minus the number of integers in the intersection of A_d and

$$\{1, 2, 3, \ldots, a_j(n) - 1\}.$$

Proof: Since the $a_n(m)$, for fixed m, satisfy the same recursion formula as the h_n, we see from (R′) that

$$a_{j+1}(n) = 2a_j(n) - a_{j-d}(n).$$

This and (13) give us

$$a_{j+1}(n) = 2a_j(n) + \{a_{-d}[a_j(n)] - 1\} \text{ for } 1 \le j < d. \tag{14}$$

Using Theorem 3, we note that the expression in braces in (14) counts the integers that are in both A_d and $\{1, 2, \ldots, a_j(n) - 1\}$. This establishes the theorem.

Theorem 4 provides a very simple procedure for calculating the $a_j(n)$ for $1 \le j \le d$. We know that $a_1(1) = 1$. Then the theorem gives us $a_j(1)$ for $1 < j \le d$. Next, $a_1(2)$ must be the smallest positive integer not among the $a_j(1)$ and the theorem gives us the remaining $a_j(2)$. Thus, one obtains the $a_j(3)$, and $a_j(4)$, etc.

Theorem 5: For $1 \le j < d$, let $g_j(m) = a_{j+1}(m) - a_j(m)$, and $G_j = \{g_j(m) \mid m \in Z^+\}$. Then $G_1, G_2, \ldots, G_{d-1}$ form a partition of Z^+.

Proof: Let Z^* be the set of positive integers that are not in A_d. For every n in Z^* there are integers m and j with $n = a_j(m)$, $m \ge 1$, and $1 \le j < d$; we let $x(n)$ be $g_j(m)$ for this m and j. Let $a_d(m) = a_m$ for m in Z^+.

Then it follows from Theorem 4 that

$$x(n) = a_{j+1}(m) - a_j(m) = a_j(m) = n \text{ for } n = 1, 2, \ldots, a_1 - 1;$$

$$x(n) = a_j(m) - 1 = n - 1 \text{ for } n = a_1 + 1, a_1 + 2, \ldots, a_2 - 1;$$

and in general that

$$x(n) = n - r \text{ for } n = a_r + 1, a_r + 2, \ldots, a_{r+1} - 1.$$

This shows that every positive integer is an $x(n)$ for exactly one n in Z^* and hence is in exactly one of the G_j, as desired.

8. BIBLIOGRAPHY

This paper is self-contained except for motivation. Related material is contained in [1], [2], and [3] and in the papers of the bibliography in [2]. It is expected to have sequels to the present paper.

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