ENUMERATION OF TRUNCATED LATIN RECTANGLES

F. W. LIGHT, JR.
229 W. Swatara Dr., P.O. Box 276, Jonestown, PA 17038

NOMENCLATURE

An $r \times k$ *rectangle* is a rectangular array of *elements* (natural numbers) with r rows and k columns. A row with no repeated element is an *R-row*. A column with no repeated element is a C-column; otherwise, it is a $\overline{C}$-column. If all rows of a rectangle are R-rows, it is an *R-rectangle*. An R-rectangle subject to no further restrictions will be called, for emphasis, *free*. One whose first row is prescribed (elements arranged in increasing numerical order) is a *normalized* R-rectangle.

An R-rectangle all of whose columns are C-columns is an *R-C-rectangle*; one whose columns are all $\overline{C}$-columns is an *R-$\overline{C}$-rectangle*. An $r \times n$ R-C-rectangle each of whose rows consists of the same n elements is a *Latin rectangle* (L-rectangle). (R-C-rectangles whose rows do not all consist of the same elements are the "truncated" L-rectangles of the title.)

ENUMERATION OF CERTAIN R-RECTANGLES

The most obvious enumerational question about L-rectangles is, probably: How many distinct normalized $r \times n$ L-rectangles are there? Denoting this number as M_n^r, we have, as in [1],

$$M_n^r = \sum_{k=0}^{n} (-1)^k \frac{\alpha_{r,n}^k}{k!} [(n-k)!]^{r-1}, \tag{1}$$

where $\alpha_{r,n}^k$ is the number of free $r \times k$ R-$\overline{C}$-rectangles that can be built up with $\overline{C}$-columns constructed from elements selected from r rows each of which consists of the elements $1, 2, \ldots, n$.

The number of free $r \times n$ L-rectangles is

$$N_n^r = \sum_{s=0}^{n} (-1)^s \binom{n}{s} [(n-s)!]^r \alpha_{r,n}^s, \tag{2}$$

since $N_n^r = n! M_n^r$.

Such formulas are effective numerically, of course, only if all the $\alpha_{r,n}^k$ are known. This is the case for $r \leq 4$, viz. ($\alpha_{r,n}^0 \equiv 1$, by definition):

$\alpha_{1,n}^k = 0$ for all $k > 0$ and all n.

$\alpha_{2,n}^k = n^{(k)}$, where $n^{(k)} = n(n-1) \ldots (n-k+1)$, a notation used throughout this report.

$\alpha_{3,n}^k = n(3n-2k)\alpha_{3,n-1}^{k-1} + 2(k-1)n(n-1)\alpha_{3,n-2}^{k-2}$, a result easily obtained by eliminating the β_i from the pair of formulas given in [1].

$\alpha_{4,n}^k$ may be found by using the 13 recurrences given in [1].

Except for $k \leq 4$ (see below), the $\alpha_{r,n}^k$ for $r > 4$ are, in general, not known.

Consider now R-C-rectangles that are not necessarily L-rectangles. Let

r = number of rows,

m = number of columns ($m \le n$)

n = number of elements available for each row (the same set of elements for each row).

$N^r_{m,n}$ = number of free R-C-rectangles with the indicated specifications.

We have

$$(3) \qquad N^r_{m,n} = \sum_{s=0}^{m} (-1)s\binom{m}{s}[(n - s)^{(m-s)}]^r \alpha^s_{r,n}.$$

Formula (3) may be derived by using the same n^r-cube that was used ([1]) to get the formula for M^r_n. In this instance, we work with only the first m of the structures of highest dimensional level (thus with stripes, if $r = 4$). Proceeding as in the earlier case, and making appropriate adjustments in the multipliers that arise (e.g., if $r = 4$, the number of k-tuples of bad cells in any m ($\ge k$) stripes is now

$$\frac{m^{(k)}}{n^{(k)}}\alpha^k_{4,n};$$

each k-tuple of bad cells combines with $[(n - k)^{(m-k)}]^3$ cells—of any kind), we get a formula for $M^r_{m,n}$ (the normalized counterpart of $N^r_{m,n}$) and finally, since $N^r_{m,n} = n^{(m)}M^r_{m,n}$, formula (3).

The free R-C-rectangles are more convenient in many respects than the normalized ones. It is immediate that there is a reciprosity between m and r:

$$(4) \qquad N^r_{m,n} = N^m_{r,n}.$$

Formula (3) may be inverted, to give:

$$(5) \qquad \alpha^m_{r,n} = \sum_{s=0}^{m} (-1)^s\binom{m}{s}[(n - s)^{(m-s)}]^r N^r_{s,n}.$$

Formulas (3) and (5) are identical, the self-inversive property being, of course, inherest in the definitions of $\alpha^m_{r,n}$ and $N^r_{m,n}$. By utilizing (4) and (5), we can find $\alpha^m_{r,n}$ for $m \le 4$, for any values of r and n. Thus, the first few terms of (2) are known for $r > 4$.

A more general formula of the sort discussed above can be given, covering cases in which some columns are C-columns and some are $\overline{C}$-columns. Let

$N^r_{m,k;n}$ = number of free R-rectangles in which:

r = total number of rows,

k = total number of columns,

m = number of C-columns (the other $k - m$ being $\overline{C}$-columns),

n = number of elements available for each row (the same set for each row).

Clearly, $m \le k \le n$.

Then

$$N^r_{m,k;n} = \binom{k}{m}\sum_{s=0}^{m}(-1)^s\binom{m}{s}[(n-k+m-s)^{(m-s)}]^r\alpha^{k-m+s}_{r,n}. \tag{6}$$

The derivation resembles that of (3), the diagram for the n^r-cube again being helpful.

A few special cases are:

If $m = 0$, we have $N^r_{0,k;n} = \alpha^k_{r,n}$.

If $m = k$, we have $N^r_{m,m;n} = N^r_{m,n}$.

If $k = n$, we have $N^r_{m,n;n}$, the number of free $r \times n$ R-rectangles each of whose rows consists of the elements 1, 2, ..., n, having m C-columns and $n - m$ $\overline{C}$-columns.

Note that $N^r_{m,k;n}$ is divisible by $k!$ (giving the number of normalized R-rectangles with the specified properties). That result is further divisible by $\binom{k}{m}$ (giving the number of normalized R-rectangles with the m C-columns preceding the $k - m$ $\overline{C}$-columns). That result is still further divisible by $\binom{n}{k}$ (giving the number of normalized R-rectangles whose C-columns start with 1, 2, ..., m in that order, and whose $\overline{C}$-columns start with $m + 1$, $m + 2$, ..., k in that order). Thus, $N^r_{m,k;n}$ is divisible by $\binom{k}{m}n^{(k)}$. For example, if $r = 2$, $k = 5$, $m = 3$, $n = 6$,

$N^2_{3,5;6} = 79{,}200$, the number of free R-rectangles;

$\frac{79{,}200}{5!} = 660$, the number of normalized R-rectangles;

$\frac{660}{\binom{5}{3}} = 66$, the number with C-columns preceding $\overline{C}$-columns;

and finally,

$\frac{66}{\binom{6}{5}} = 11$, the number with 3 C-columns headed by 1, 2, 3 and 2 $\overline{C}$-columns headed by 4, 5 in that order, as may be verified easily by direct count.

REFERENCE

1. F. W. Light, Jr., "A Procedure for the Enumeration of $4 \times n$ Latin Rectangles," *The Fibonacci Quarterly* 11, No. 3 (1973):241-246.

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