GENERATING FUNCTIONS OF CENTRAL VALUES IN GENERALIZED PASCAL TRIANGLES

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1. INTRODUCTION

In this paper we shall examine the generating functions of the central (maximal) values in Pascal's binomial and trinomial triangles. We shall compare the generating functions to the generating functions obtained from partition sums in Pascal's triangles.

Generalized Pascal triangles arise from the multinomial coefficients obtained by the expansion of

$$(1 + x + x^2 + \cdots + x^{j-1}) \ , \quad j \geq 2, \ n \geq 0,$$

where "n" denotes the row in each triangle. For $j = 2$, the binomial coefficients give rise to the following triangle:

1				
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1
etc.				

For $j = 3$, the trinomial coefficients produce the following triangle:

1						
1	1	1				
1	2	3	2	1		
1	3	6	7	6	3	1

The partition sums are defined

$$S(n,j,k,r) = \sum_{i=0}^{M} \left[\!\left[\begin{matrix} n \\ r+ik \end{matrix} \right]\!\right]_j; \quad 0 \leq r \leq k-1,$$

where

$$M = \left[\frac{(j-1)n - r}{k}\right],$$

[ ] denoting the greatest integer function. To clarify, we give a numerical example. Consider $S(6,3,4,2)$. This denotes the partition sums in the sixth row of the trinomial triangle in which every fourth element is added, beginning with the second column. The $S(6,3,4,2) = 15 + 45 + 1 = 61$. (Conventionally, the column of 1's at the far left is the 0th column and the top row is the 0th row.)

In the nth row of the j-nomial triangle the sum of the elements is j^n. This is expressed by

$$S(n,j,k,0) + S(n,j,k,1) + \cdots + S(n,j,k,k-1) = j^n.$$

Let

$$S(n,j,k,0) = (j^n + A_n)/k$$

$$S(n,j,k,1) = (j^n + B_n)/k \ldots$$
$$S(n,j,k,k-1) = (j^n + Z_n)/k.$$

Since $S(0,j,k,0) = 1$,

$$S(0,j,k,1) = 0 \ldots S(0,j,k,k-1) = 0,$$

we can solve for A_0, B_0, ..., Z_0 to get $A_0 = k - 1$, $B_0 = -1$, ..., $Z_0 = -1$.

Now a departure table can be formed with A_0, B_0, ..., Z_0 as the 0th row. The term "departure" refers to the quantities, A_n, B_n, ..., Z_n that depart from the average value j^n/k. Pascal's rule of addition is the simplest method for finding the successive rows in each departure table. The departure tables for 5 and 10 partitions in the binomial triangle appear below. Notice the appearance of Fibonacci and Lucas numbers.

Table 1

SUMS OF FIVE PARTITIONS IN THE BINOMIAL TRIANGLE

4	-1	-1	-1	-1
3	3	-2	-2	-2
1	6	1	-4	-4
-3	7	7	-3	-8
-11	5	14	4	-11
-22	-7	18	18	-7

Table 2

SUMS OF TEN PARTITIONS IN THE BINOMIAL TRIANGLE

9	-1	-1	-1	-1	-1	-1	-1	-1	-1
8	8	-2	-2	-2	-2	-2	-2	-2	-2
6	16	6	-4	-4	-4	-4	-4	-4	-4
2	22	22	2	-8	-8	-8	-8	-8	-8
-6	24	44	24	-6	-16	-16	-16	-16	-16
-22	18	68	68	18	-22	-32	-32	-32	-32

The primary purpose of this paper is to show that the limit of the generating functions for the $(H - L)/k$ sequences is precisely the generating functions for the central values in the rows of the binomial and trinomial triangles. The $(H - L)/k$ sequences are obtained from the difference of the maximum and minimum value sequences in a departure table, divided by k, where k denotes the number of partitions.

2. GENERATING FUNCTIONS OF THE $(H - L)/k$ SEQUENCES IN THE BINOMIAL TRIANGLE

Table 3 is a table of the $(H - L)/k$ sequences for $k = 3$ to $k = 15$ partitions.

The generating function of the maximum values in the binomial triangle is

$$\frac{1}{\sqrt{1 - 4x^2}}\left(\frac{1 + 2x - \sqrt{1 - 4x^2}}{2x}\right).$$

We shall examine this and show it to be the limit of the generating functions of the $(H - L)/k$ sequences.

Table 3

$(H - L)/k$ SEQUENCES FOR $k = 3$ TO $k = 15$

k = 3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	2	2	2	2	2	2	2	2	2	2	2
1	2	3	3	3	3	3	3	3	3	3	3	3
1	4	5	6	6	6	6	6	6	6	6	6	6
1	4	8	9	10	10	10	10	10	10	10	10	10
1	8	13	18	19	20	20	20	20	20	20	20	20
1	8	21	27	33	34	35	35	35	35	35	35	35
1	16	34	54	61	68	69	70	70	70	70	70	70
1	16	55	81	108	116	124	125	126	126	126	126	126
1	32	89	162	197	232	241	250	251	252	252	252	252
1	32	144	243	352	396	440	450	460	461	462	462	462
1	64	233	496	638	792	846	900	911	922	923	924	924
1	64	377	729	1145	1352	1560	1625	1690	1702	1714	1715	1716
1	128	610	1458	2069	2704	2977	3250	3327	3404	3417	3430	3431

Consider the relation $S_{n+2} = S_{n+1} - x^2 S_n$, expressed by the equation

$$K^2 - K + x^2 = 0.$$

The two roots are

$$K_1 = \frac{1 + \sqrt{1 - 4x^2}}{2} \quad \text{and} \quad K_2 = \frac{1 - \sqrt{1 - 4x^2}}{2}, \quad K_1 > K_2.$$

Therefore,

$$\lim_{n \to \infty} \frac{S_{n+1}}{S_n} = K_1 = L$$

by Gauss's theorem that the limit is the root of the maximum modulus.

The generating functions for the odd partitions, $k = 2m + 1$, were found to have the form

$$\frac{S_{m-1}}{S_m - xS_{m-1}}.$$

The generating functions for the even partitions, $k = 2m$, were found to have the form

$$\frac{S_{m-1} + S_{m-2}}{S_m - x^2 S_{m-2}}.$$

We show these two forms have the same limit.

$$\lim_{n \to \infty} \frac{S_{n-1}}{S_n - xS_{n-1}} = \frac{\dfrac{S_{n-1}}{S_{n-1}}}{\dfrac{S_n}{S_{n-1}} - \dfrac{xS_{n-1}}{S_{n-1}}} = \frac{1}{L - x}$$

$$\lim_{n\to\infty} \frac{S_{n-1} + S_{n-2}}{S_n - x^2 S_{n-2}} = \frac{\dfrac{S_{n-1}}{S_{n-2}} + \dfrac{xS_{n-2}}{S_{n-2}}}{\dfrac{S_n}{S_{n-2}} - \dfrac{x^2 S_{n-2}}{S_{n-2}}} = \frac{L + x}{L^2 - x^2} = \frac{1}{L - x}$$

where

$$\frac{1}{L - x} = \frac{1}{\dfrac{1 + \sqrt{1 - 4x^2} - 2x}{2}} = \frac{2}{1 - 2x + \sqrt{1 - 4x^2}}$$

$$= \frac{1}{\sqrt{1 - 4x^2}}\left(\frac{1 + 2x - \sqrt{1 - 4x^2}}{2x}\right).$$

We pause now to consider the generating function for

$$1 + 2x + 6x^2 + 20x^3 + 70x^4 + \cdots + \binom{2n}{n}x^n + \cdots = \frac{1}{\sqrt{1 - 4x}},$$

(see [1], p. 41).

Now the Catalan number $\dfrac{1}{n + 1}\dbinom{2n}{n}$ generating function is

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

Thus,

$$\frac{1}{\sqrt{1 - 4x}}\left(\frac{1 - \sqrt{1 - 4x}}{2x}\right) = 1 + 3x + 10x^2 + 35x^3 + \cdots,$$

(see [2], p. 8).

We observe the following relationship between these two series:

$$(1 + 2x + 6x^2 + 20x^3 + 70x^4 + \cdots - 1)/2x$$

$$= \frac{2x(1 + 3x + 10x^2 + 35x^3 + \cdots}{2x} = \frac{1}{\sqrt{1 - 4x}}\left(\frac{1 - \sqrt{1 - 4x}}{2x}\right).$$

Next we wish to blend these two series. Replace x with x^2.

$$\frac{1}{\sqrt{1 - 4x^2}} = 1 + 2x^2 + 6x^4 + 20x^6 + 70x^8 + \cdots .$$

We multiply the latter by x, after replacing x with x^2.

$$x \quad \frac{1}{\sqrt{1 - 4x^2}}\left(\frac{1 - \sqrt{1 - 4x^2}}{2x^2}\right) = x + 3x^3 + 10x^5 + 35x^7 + \cdots .$$

Therefore, the generating function for the blend,

$$1 + x + 2x^2 + 3x^3 + 6x^4 + 10x^5 + 20x^6 + \cdots$$

is

$$\frac{1}{\sqrt{1 - 4x^2}}\left(1 + \frac{1 - \sqrt{1 - 4x^2}}{2x}\right) = \frac{1}{\sqrt{1 - 4x^2}}\left(\frac{1 + 2x - \sqrt{1 - 4x^2}}{2x}\right)$$

which is precisely the value of $\frac{1}{L - x}$. Thus, we see that the limit of the generating functions for the $(H - L)/k$ sequences is precisely the generating function for the maximum values in the rows of the binomial triangle.

3. GENERATING FUNCTIONS OF THE $(H - L)/k$ SEQUENCES IN THE TRINOMIAL TRIANGLE

Table 4 exhibits the $(H - L)/k$ sequences for $k = 4$ to $k = 16$ partitions. The generating function of the maximum values in the trinomial triangle is

$$1/\sqrt{1 - 2x - 3x^2}.$$

Table 4

$(H - L)/k$ SEQUENCES FOR $k = 4$ TO $k = 16$

k = 4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	3	3	3	3	3	3	3	3	3	3	3
1	3	5	6	7	7	7	7	7	7	7	7	7
1	5	11	14	17	18	19	19	19	19	19	19	19
1	8	21	31	41	45	49	50	51	51	51	51	51
1	13	43	70	99	114	129	134	139	140	141	141	141
1	21	85	157	239	288	337	358	379	385	391	392	393

Consider the relation $F_{n+2} = F_{n+1} + F_n$, which is expressed by the equation

$$x^2 - x - 1 = 0.$$

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = L$$

and

$$\frac{F_{n+2}}{F_{n+1}} = 1 + \frac{1}{\dfrac{F_{n+1}}{F_n}},$$

thus

$$L = 1 + \frac{1}{L},$$

so

$$L^2 = L + 1,$$

or

$$L^2 - L - 1 = 0.$$

Next consider the relation $S_{n+3} = S_{n+2} - xS_{n+1} + x^3 S_n$, expressed by the equation

$$K^3 - K^2 + xK - x^3 = 0,$$

or in factored form

$$(K - x)\left(K^2 - (1 - x)K + x^2\right) = 0.$$

$$\lim_{n\to\infty} \frac{S_{n+1}}{S_n} = L,$$

which is the root of the maximum modulus by Gauss's theorem. Further,

$$\lim_{n\to\infty} \frac{S_n}{S_{n+1} - x^2 S_{n-1}} = \frac{1}{\sqrt{1 - 2x - 3x^2}},$$

which is the generating function of the maximum values of the trinomial triangle.

Assume

$$\frac{S_n}{S_{n+1} - x^2 S_{n-1}} = \frac{\dfrac{S_n}{S_{n-1}}}{\dfrac{S_{n+1}}{S_{n-1}} - x^2} = \frac{L}{L^2 - x^2},$$

where

$$L = \lim_{n\to\infty} \frac{S_n}{S_{n-1}}$$

and

$$L^2 = \lim_{n\to\infty} \frac{S_{n+1}}{S_n} \cdot \frac{S_n}{S_{n-1}}.$$

The roots of

$$K^3 - K^2 + xK - x^3 = 0$$

are

$$x, \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2},$$

and

$$\frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2}.$$

The dominant root is $\dfrac{1 - x + \sqrt{1 - 2x - 3x^2}}{2}$, which is L. Thus,

$$L^2 = \frac{(1 - x)^2 - 2x^2 + (1 - x)\sqrt{1 - 2x - 3x^2}}{2}$$

and

$$L^2 - x^2 = \frac{(1 - x)^2 - 4x^2 + (1 - x)\sqrt{1 - 2x - 3x^2}}{2}.$$

Therefore,

$$\begin{aligned}\frac{L}{L^2 - x^2} &= \frac{1 - x + \sqrt{1 - 2x - 3x^2}}{(1 - x)^2 - 4x^2 + (1 - x)\sqrt{1 - 2x - 3x^2}} \\ &= \frac{1 - x + \sqrt{1 - 2x - 3x^2}}{(1 - 2x - 3x^2) + (1 - x)\sqrt{1 - 2x - 3x^2}} \\ &= \frac{1}{\sqrt{1 - 2x - 3x^2}}.\end{aligned}$$

The generating functions for the odd cases were found to have the form

$$\frac{S_n(x)}{S_{n+1}(x) - x^2S_{n-1}(x)}.$$

The polynomials S_n with the recurrence $S_{n+3} = S_{n+2} - xS_{n+1} + x^3S_n$ are listed as follows:

$$S_0 = 0$$
$$S_1 = 1$$
$$S_2 = 1$$
$$S_3 = 1 - x$$
$$S_4 = 1 - 2x + x^3$$
$$S_5 = 1 - 3x + x^2 + 2x^3$$
$$S_6 = 1 - 4x + 3x^2 + 3x^3 - 2x^4$$
$$S_7 = 1 - 5x + 6x^2 + 3x^3 - 6x^4 + x^6$$

etc.

Thus, the generating functions for $N = 2n + 1$ are as follows:

$$N = 5 \text{ is } \frac{S_2}{S_3 - x^2S_1} = \frac{1}{1 - x - x^2}$$

$$N = 7 \text{ is } \frac{S_3}{S_4 - x^2S_2} = \frac{1 - x}{1 - 2x - x^2 - x^3}$$

$$N = 9 \text{ is } \frac{S_4}{S_5 - x^2S_3} = \frac{1 - 2x + x^3}{1 - 3x + 3x^3}$$

$$N = 11 \text{ is } \frac{S_5}{S_6 - x^2S_4} = \frac{1 - 3x + x^2 + 2x^3}{1 - 4x + 2x^2 + 5x^3 - 2x^4 - x^5}$$

$$N = 13 \text{ is } \frac{S_6}{S_7 - x^2S_5} = \frac{1 - 4x + 3x^2 + 3x^3 - 2x^4}{1 - 5x + 5x^2 + 6x^3 - 7x^4 - 2x^5 + x^6}$$

$$N = 15 \text{ is } \frac{S_7}{S_8 - x^2S_6} = \frac{1 - 5x + 6x^2 + 3x^3 - 6x^4 + x^6}{1 - 6x + 9x^2 + 5x^3 - 15x^4 + 5x^6}$$

Before the generating functions for the even cases are given, the Lucas, $L_n(x)$, and Fibonacci, $F_n(x)$, polynomials for the factor $K^2 - (1 - x)K + x^2$ will be derived. The Lucas and Fibonacci polynomials are defined:

$$L_n(x) = a^n(x) + b^n(x)$$
$$F_n(x) = a^n(x) - b^n(x)/a(x) - b(x)$$

where a and b are the roots of the polynomial equation

$$K^2 - A(x)K + B(x) = 0.$$

The recurrence relation for the Lucas polynomials is

$$L_{n+2}(x) = (1 - x)L_{n+1}(x) - x^2L_n(x).$$

The polynomials are

$$L_0 = 2$$
$$L_1 = 1 - x$$
$$L_2 = 1 - 2x - x^2$$
$$L_3 = 1 - 3x + 2x^3$$
$$L_4 = 1 - 4x + 2x^2 + 4x^3 - x^4$$
$$L_5 = 1 - 5x + 5x^2 + 5x^3 - 5x^4 - x^5$$

etc.

The recurrence relation for the Fibonacci polynomials is

$$F_{n+2}(x) = (1 - x)F_{n+1}(x) - x^2F_n(x).$$

The polynomials are

$$F_0 = 0$$
$$F_1 = 1$$
$$F_2 = 1 - x$$
$$F_3 = 1 - 2x$$
$$F_4 = 1 - 3x + x^2 + x^3$$
$$F_5 = 1 - 4x + 3x^2 - 2x^3 - x^4$$

etc.

The generating functions for $N = 4n$ were found to have the form

$$\frac{F_n}{L_n}$$

and the generating functions for $N = 4n + 2$ were found to be

$$\frac{F_n - x^2F_{n-1}}{L_n - x^2L_{n-1}}.$$

They are listed below.

$$N = 4 \text{ is } \frac{F_1}{L_1} = \frac{1}{1 - x}$$

$$N = 6 \text{ is } \frac{F_1 - x^2F_0}{L_1 - x^2L_0} = \frac{1}{1 - x - 2x^2}$$

$$N = 8 \text{ is } \frac{F_2}{L_2} = \frac{1 - x}{1 - 2x - x^2}$$

$$N = 10 \text{ is } \frac{F_2 - x^2F_1}{L_2 - x^2L_1} = \frac{1 - x - x^2}{1 - 2x - 2x^2 + x^3}$$

$$N = 12 \text{ is } \frac{F_3}{L_3} = \frac{1 - 2x}{1 - 3x + 2x^3}$$

$$N = 14 \text{ is } \frac{F_3 - x^2F_2}{L_3 - x^2L_2} = \frac{1 - 2x - x^2 + x^3}{1 - 3x - x^2 + 4x^3 + x^4}$$

$$N = 16 \text{ is } \frac{F_4}{L_4} = \frac{1 - 3x + x^2 + x^3}{1 - 4x + 2x^2 + 4x^3 - x^4}$$

Lastly, we show

$$\lim_{n\to\infty} \frac{F_n}{L_n} = \frac{1}{\sqrt{1 - 2x - 3x^2}} \quad \text{and} \quad \lim_{n\to\infty} \frac{F_n - x^2F_{n-1}}{L_n - x^2L_{n-1}} = \frac{1}{\sqrt{1 - 2x - 3x^2}}.$$

We recall that the equation

$$K^2 - (1 - x)K + x^2 = 0$$

has roots

$$K_1 = \frac{(1 - x) + \sqrt{(1 - x)^2 - 4x^2}}{2} \quad \text{and} \quad K_2 = \frac{(1 - x) - \sqrt{(1 - x)^2 - 4x^2}}{2}.$$

We define

$$F_n = \frac{K_1^n - K_2^n}{K_1 - K_2}$$

and

$$L_n = K_1^n + K_2^n.$$

Note that

$$K_1 - K_2 = \sqrt{1 - 2x - 3x^2}.$$

Thus,

$$\frac{F_n}{L_n} = \frac{K_1^n - K_2^n}{(K_1 - K_2)(K_1^n + K_2^n)} = \frac{1 - \left(\dfrac{K_2}{K_1}\right)^n}{1 + \left(\dfrac{K_2}{K_1}\right)^n (K_1 - K_2)}$$

Now, since $K_1 > K_2$,

$$\lim_{n\to\infty} \frac{1 - \left(\dfrac{K_2}{K_1}\right)^n}{1 + \left(\dfrac{K_2}{K_1}\right)^n (K_1 - K_2)} = \frac{1}{\sqrt{1 - 2x - 3x^2}} = L.$$

We use this result to prove the second limit $= L$.

$$\lim_{n\to\infty} \frac{F_n - x^2F_{n-1}}{L_n - x^2L_{n-1}} = \frac{\dfrac{F_n}{L_{n-1}} - x^2\dfrac{F_{n-1}}{L_{n-1}}}{\dfrac{L_n}{L_{n-1}} - x^2\dfrac{L_{n-1}}{L_{n-1}}} = \frac{L^2 - x^2L}{L - x^2} = L,$$

since

$$\frac{F_n}{L_{n-1}} = \frac{F_n}{F_{n-1}}\,\frac{F_{n-1}}{L_{n-1}} = L^2.$$

4. GENERATING FUNCTIONS OF THE $(H - L)/k$ SEQUENCES IN A MULTINOMIAL TRIANGLE

We challenge the reader to find the generating functions of the $(H - L)/k$ sequences in the quadrinomial triangle. We surmise that the limits would be the generating functions of the central values in Pascal's quadrinomial triangle.

REFERENCES

1. John L. Brown, Jr., & V. E. Hoggatt, Jr., "A Primer for the Fibonacci Numbers, Part XVI: The Central Column Sequence," *The Fibonacci Quarterly* 16, No. 1 (1978):41.
2. Michel Y. Rondeau, "The Generating Functions for the Vertical Columns of $(N + 1)$-Nomial Triangles" (Master's thesis, San Jose State University, San Jose, California, May 1978).
3. Claudia R. Smith, "Sums of Partition Sets in the Rows of Generalized Pascal's Triangles" (Master's thesis, San Jose State University, San Jose, California, May 1978).

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SOLUTION OF $\binom{y+1}{x} = \binom{y}{x+1}$ IN TERMS OF FIBONACCI NUMBERS

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In [2, pp. 262-263] we solved the Diophantine equation $\binom{y+1}{x} = \binom{y}{x+1}$ and found that (x,y) is a solution iff for some $n \geq 0$,

$$(x + 1, y + 1) = \left(\sum_{k=0}^{n} f(4k + 1),\ \sum_{k=0}^{n} f(4k + 3)\right),$$

where

$$f(0) = 0,\ f(1) = 1,\ f(n + 2) = f(n) + f(n + 1).$$

We show here that (x,y) is a solution iff for some $n \geq 0$,

$$(x + 1, y + 1) = \big(f(2n + 1)f(2n + 2),\ f(2n + 2)f(2n + 3)\big),$$

incidentally deriving the identities

$$f(2n + 1)f(2n + 2) = \sum_{k=0}^{n} f(4k + 1),$$

$$f(2n + 2)f(2n + 3) = \sum_{k=0}^{n} f(4k + 3).$$