FIBONACCI NUMBERS

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The purpose of this paper is to derive a few relations involving Fibonacci numbers; these numbers are defined according to the expressions

$$f_{n+1} = f_n + f_{n-1},\ f_0 = 0,\ f_1 = 1$$

due to Girard [1]. They can also be obtained from a known [2] matrix representation that we rederive in Part II. In Part III we obtain the sum of two infinite series and some recurrence relations.

PART I: HISTORICAL NOTE

The sequence of integers $\{f_n\}$ was discovered by Leonardo Pisano [3, 4], in his *Liber Abacci*, as the solution to a hypothetical problem concerning the breeding of rabbits; in this problem, Pisano admitted that the rabbits never die, that each month every pair begets a new pair that becomes productive at the age of two months. The experiment begins in the first month with a newborn pair. Fibonacci numbers occur in many different areas. In geometry, for instance, in Euclid's golden section problem where the number $\frac{1}{2}(\sqrt{5} - 1)$ appears. In the botanical phenomenon called phyllotaxis, where it is well known that in some trees the leaves are disposed in the spirals according to the Fibonacci sequence

$$\frac{1}{1},\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{5},\ \ldots,\ \frac{f_n}{f_{n+1}}$$

that results from the expansion of $\frac{1}{2}(\sqrt{5} - 1)$ in continued fractions. It is also known that in the sunflower the number of spirals usually present are the Fibonacci numbers 34 and 55; in the giant sunflower they are 55 and 89, and recent experiments have reported that sunflowers of 89 and 144 as well as 144 and 233 spirals also exist. These are all Fibonacci numbers.

PART II: THEORY

Consider the numbers f_1', $k = 0, 1, 2, \ldots$, defined by

$$\begin{pmatrix} f_{k+1}' & f_k' \\ f_k' & f_{k-1}' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k. \tag{2.1}$$

For $k = 1$, we have $f_0' = f_0$, $f_1' = f_1$, and $f_2' = f_2$. Let us suppose that $f_n' = f_n$ is valid for arbitrary n. It is easily seen from (2.1) that $f_n' = f_n$ is also valid for $n \quad n + 1$, since we have from (2.1) that

$$f_{n+2}' = f_{n+1} + f_n = f_{n+2};\ f_{n+1}' = f_n + f_{n-1} = f_{n+1}.$$

We see then that (2.1) defines the Fibonacci numbers f_n.

Define the matrices $F(n)$ and A according to the following expressions:

$$F(n) = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = A^n; \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.2}$$

It is easily proved that the above equation contains Lucas' definition of Fibonacci numbers:

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]; \tag{2.3}$$

in fact, the eigenvalues of A are $\lambda_1 = \frac{1}{2}(1+\sqrt{5})$ and $\lambda_2 = \frac{1}{2}(1-\sqrt{5})$. We see therefore that the matrix that diagonalizes A is given by

$$U = \begin{pmatrix} \alpha_1\lambda_1 & \alpha_2\lambda_2 \\ \alpha_1 & \alpha_2 \end{pmatrix}, \text{ where } \alpha_i = (1+\lambda_i^2)^{-1/2}, \tag{2.4}$$

$$U^{-1}AU = \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

We have then, from (2.2),

$$F(n) = U\Lambda^n U^{-1}, \tag{2.5}$$

which explicitly reads as:

$$\begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} & \lambda_1^n - \lambda_2^n \\ \lambda_1^n - \lambda_2^n & \lambda_1^{n-1} - \lambda_2^{n-1} \end{pmatrix}$$

PART III: SERIES AND RECURRENCE RELATIONS

From (2.2), we write the following expression:

$$\sum_1^\infty \frac{1}{n!}F(n) = e^A - 1, \tag{3.1}$$

from which we infer that

$$\sum_1^\infty \frac{1}{n!}U^{-1}F(n)U = e^\Lambda - 1. \tag{3.2}$$

The matrix elements are given by:

$$[U^{-1}F(n)U]_{11} = \frac{1}{2}(f_{n+1} + f_{n-1}) + \frac{\sqrt{5}}{2}f_n = \alpha^n; \tag{3.3}$$

$$[U^{-1}F(n)U]_{12} = -[U^{-1}F(n)U]_{21} = \frac{\sqrt{5}}{2}(f_{n+1} - f_n - f_{n-1}) = 0;$$

$$[U^{-1}F(n)U]_{22} = \frac{1}{2}(f_{n+1} + f_{n-1}) - \frac{\sqrt{5}}{2}f_n = \beta^n.$$

From (3.1), the following series are derived:

$$\sum_0^\infty \frac{1}{n!} f_n = \frac{2e^{1/2}}{\sqrt{5}} \sinh\left(\frac{\sqrt{5}}{2}\right) \tag{3.4}$$

$$\sum_0^\infty \frac{1}{n!}(f_{n+1} + f_{n-1}) = 2e^{1/2} \cosh\left(\frac{\sqrt{5}}{2}\right),$$

where we extended Fibonacci numbers to negative values according to

$$f_{-n} = (-1)^{n+1} f_n.$$

We now set $A = 1 + B$ in (2.2) to obtain

$$F(n) = \sum_0^n \binom{n}{k} B^k. \tag{3.5}$$

B^k can be easily evaluated if we use Cauchy's integral

$$B^k = (2\pi i)^{-1} \int (dZ) Z^k (Z - B)^{-1}.$$

B^k is given by

$$B^k = F(k)^{-1} = \begin{pmatrix} f_{k-1} & -f_k \\ -f_k & f_{k+1} \end{pmatrix} (-1)^k. \tag{3.6}$$

Therefore, we have the following recurrence relations that also define Fibonacci numbers if we add to them the appropriate boundary conditions

$$f_0 = 0,\ f_1 = 1:$$

$$f_{n\pm1} = \sum_0^n (-1)^k \binom{n}{k} f_{k\pm1} \tag{3.7}$$

$$f_n = \sum_0^n (-1)^{k+1} \binom{n}{k} f_k.$$

If we multiply (2.2) by $(-1)^n F(n)^{-1}$, we obtain the following orthogonality relations:

$$\sum_0^n (-1)^k \binom{n}{k} f_{n+k\pm1} = (-1)^n \tag{3.8}$$

$$\sum_0^n (-1)^k \binom{n}{k} f_{n+k} = 0.$$

Many important relations can be easily obtained from (2.2), and we just list a few of them.

The determinant of (2.2) gives

$$f_{n+1}f_{n-1} - f_n^2 = (-1)^n.$$

Setting $n = j + k$ and $A^n = A^j A^k$ in (2.2) gives the following well-known recurrence relations:

(3.9) $$f_{j+k\pm 1} = f_{j\pm 1} f_{k\pm 1} + f_j f_k;$$
$$f_{j+k} = f_{j+1} f_k + f_j f_{k+1}.$$

From the above, or from $F(np) = F(n)^p$, we are also able to obtain other familiar expressions such as:

(3.10) $$f_{2n\pm 1} = f_n^2 + f_{n\pm 1}^2;$$
$$\frac{f_{2n}}{f_n} = f_{n+1} + f_{n-1}$$
$$f_{3n} = f_{n+1}^3 + f_n^3 - f_{n-1}^3;$$
$$\frac{f_{3n}}{f_n} = 2f_{n+1}^2 + f_n^2 + f_{n+1} f_{n-1}.$$

REFERENCES

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A NOTE ON BASIC M-TUPLES

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Definition 1: A set of integers $\{b_i\}_{i\geq 1}$ will be called a base for the set of all integers, whenever every integer n can be expressed uniquely in the form

$$n = \sum_{i=1}^{\infty} a_i b_i, \text{ where } a_i = 0 \text{ or } 1 \quad \text{and} \quad \sum_{i=1}^{\infty} a_i < \infty.$$

Now, a sequence $\{d_i\}_{i\geq 1}$ of odd numbers will be called basic whenever the sequence $\{d_i 2^{i-1}\}_{i\geq 1}$ is a base. If the sequence $\{d_i\}_{i\geq 1}$ of odd integers is such that $d_{i+s} = d_i$ for all i's, then the sequence is said to be periodic mod s and is denoted by $\{d_1, d_2, d_3, \ldots, d_s\}$. In reference [2], I have obtained some results concerning nonbasic sequence with periodicity mod 3 or nonbasic triples. In this paper, we are concerned with basic sequence.

Theorem 1: A necessary and sufficient condition for the sequence $\{d_i\}_{i\geq 1}$ of odd integers, which is periodic mod s, to be basic is that