GROWTH TYPES OF FIBONACCI AND MARKOFF*

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1. PRELIMINARY REMARKS

The pattern of Fibonaccian growth in pure and applied mathematics is well known and seemingly ubiquitous. In recent work of the author (see [1]), a generalization of this pattern emerged where the "linear" growth of Fibonacci type is replaced by a "tree" growth which might appropriately be called the "Markoff type." There are many instances where tree-growth is used for number-theoretic functions (for a recent example, see [4]). What is different here is the application of the tree to (noncommutative) strings of symbols. This, paradoxically, makes for a simpler device but one with applications to many different fields.

The use of the "Markoff" designation requires some clarification. We refer to A. A. Markoff (1856-1922), the number-theorist. He was also the probabilitist (with the name customarily spelled "Markov" in this context), but the growth type we desire is *nonrandom* and strictly a consequence of his number-theoretic work. To compound the confusion, he had a lesser known brother, V. A. Markoff (also a number-theorist), and a very famous son, the logician A. A. Markov (still alive today).

2. SEMIGROUP

We consider S_2 a free semigroup consisting of strings of symbols in A and B (including "$_1$" the null symbol) to form *words* $w = w(A,B)$. If the word w has a symbols A and b symbols B (for $a \geq 0$, $b \geq 0$), then we say word w has *coordinates* $\{a, b\}$. For instance, some coordinates and words are

$$\begin{array}{cccccc} \{0,\ 0\}, & \{1,\ 0\}, & \{0,\ 1\}, & \{1,\ 1\}, & \{1,\ 1\}, & \{4,\ 2\}, \\ 1, & A, & B, & AB, & BA, & AAABAB, \text{ etc.} \end{array}$$

Of course, distinct words (e.g., AB and BA) may have the same coordinates. Naturally, we abbreviate $AAABAB$ as A^3BAB, etc.

We also introduce the concept of *equivalence*. Two words of S_2 are said to be equivalent if they are cyclic permutations of one another including the trivial (identity) permutation. This is denoted by "$\sim$". Thus,

$$w_1(A,B)w_2(A,B) \sim w_2(A,B)w_1(A,B).$$
$$ABAA \sim ABAA \sim BAAA \sim AAAB \sim \cdots$$

Equivalent words have the same coordinates, of course (but not conversely, $ABAB$ and $AABB$ have coordinates $\{2, 2\}$).

Actually $w_1 \sim w_2$ means $Tw_1 = w_2T$ (for $T \in S_2$), and for computational purposes it might be convenient to do computations inside the *free group* by writing $w_1 = T^{-1}w_2T$. In principle, however, growth requires only a semigroup. We also need the symbol when we have multiple equivalence

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$$(w_1, w_2, \ldots) \sim (w_1', w_2', \ldots) \Longleftrightarrow Tw_1 = w_1'T,\ Tw_2 = w_2'T, \ldots$$

for the same T in each case.

3. TYPES OF GROWTH

Fibonaccian growth suggests the sequence

$$(f_{-2} = 1,\ f_{-1} = 0),\ f_0 = 1,\ f_1 = 1,\ f_2 = 2,\ \ldots,\ f_{n+1} = f_{n-1} + f_n .$$

If we start with A and B instead of f_0 and f_1 we have a sequence of strings, $w_n(A,B)$

$$w_0 = A,\ w_1 = B,\ w_2 = AB,\ \ldots,\ w_{n+1} = w_{n-1}w_n .$$

To list a few strings with coordinates

$$\begin{array}{cccccc} \{1,\ 0\}, & \{0,\ 1\}, & \{1,\ 1\}, & \{1,\ 2\}, & \{2,\ 3\}, & \\ A, & B, & AB, & BAB, & ABBAB, & \ldots \end{array}$$

Clearly $w_n(A,B)$ has the coordinates $\{f_{n-2},\ f_{n-1}\}$.

Here we have used the strings $w_n(A,B)$ instead of f_n but the progression is still linearly ordered:

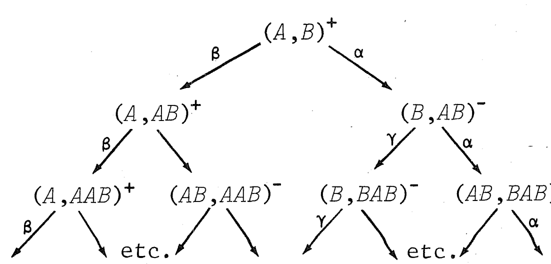
$$\cdots \to (w_{n-1}, w_n) \to (w_n, w_{n-1}w_n) \to \cdots \quad \text{(Fibonacci type)}.$$

We now consider a generalization of this growth where the ordering is not linear but tree-like,

$$\begin{tikzcd}[column sep=small] & (w', w'') \arrow[dl] \arrow[dr] & \\ (w', w'w'') & & (w'', w'w'') \end{tikzcd} \qquad \text{(Markoff type)}.$$

Thus, once $w(=w'w'')$ is formed, we have the choice of dropping w' (Fibonacci again) or dropping w''.

We illustrate the *Markoff tree* generated by starting with the pair (A,B). (The "+" and "-" signs are explained in Section 4 below).



There are 2^{n-1} possible pairs on the nth level.

The reader can easily recognize Fibonaccian growth on the extreme right diagonal (α)

$$A,\ B,\ AB,\ BAB,\ ABBAB,\ \ldots$$

On the extreme left diagonal (β), we see the simpler growth

$$B,\ AB,\ AAB,\ AAAB,\ \ldots$$

This may seem asymmetrical, but a parallel diagonal (γ) gives

$$B,\ AB,\ BAB,\ BBAB,\ \ldots$$

which is equivalent (with the same "T" = B) to

$$B,\ BA,\ BBA,\ BBBA,\ \ldots$$

4. EUCLIDEAN PARTITION

If we look at the words in the Markoff tree (in Section 3), we see that they have coordinates as follows:

$$(\{1,\ 0\},\ \{0,\ 1\})^+$$

$$(\{1,\ 0\},\ \{1,\ 1\})^+ \qquad\qquad (\{0,\ 1\},\ \{1,\ 1\})^-$$

$$(\{1,\ 0\},\ \{2,\ 1\})^+, \quad (\{1,\ 1\},\ \{2,\ 1\})^- \quad (\{0,\ 1\},\ \{1,\ 2\})^- \quad (\{1,\ 1\},\ \{1,\ 2\})^+$$

In general, a pair (w_1, w_2) has the coordinates

$$(\{a_1,\ b_1\},\ \{a_2,\ b_2\}) \quad \text{where} \quad a_1b_2 - a_2b_1 = \pm 1.$$

(The "+" and "-" designations give this sign in Section 3 and above.) We can prove an even stronger result if we introduce a definition:

Let a, a', a'', b, b', b'' all be ≥ 0, then we say

$$(a,b) = (a',b') + (a'',b'')$$

is a *euclidean partition* exactly when $a'b'' - b'a'' = +1$. Then every such (a,b) has a euclidean partition if $ab > 0$ by virtue of the *euclidean algorithm* by the solvability of

$$ax - by = \pm 1, \quad (0 \leq x < b,\ 0 \leq y < a).$$

For +1, $(x,y) = (a'',b'')$; for -1, $(x,y) = (a',b')$. Clearly any (a,b) can be ultimately partitioned to (0,1) and (1,0). For instance, if we start with (5,7), we have:

$$\begin{aligned}
&(5,7) = (3,4) + (2,3), \quad && (3,4) = (1,1) + (2,3),\\
&(2,3) = (1,1) + (1,2), \quad && (1,2) = (1,1) + (0,1),\\
&(1,1) = (1,0) + (0,1).
\end{aligned}$$

We now see, generally, that if (w',w'') is in the Markoff tree and $w = w'w''$ with $\{a',\ b'\}$, $\{a'',\ b''\}$, and $\{a,\ b\}$ the coordinates of w', w'', and w (respectively), then we write

$$(w',w'')^+ \Rightarrow (a,b) = (a',b') + (a'',b'')$$

$$(w',w'')^- \Rightarrow (a,b) = (a'',b'') + (a',b'),$$

as euclidean partitions in each case. The property is preserved in the Markoff tree, *so every* $\{a,\ b\}$ with gcd $(a,b) = 1$ (*and* $a \geq 0$, $b \geq 0$) *is represented as the coordinate of some word in the Markoff tree.*

We shall next see how words in the Markoff tree are composed by euclidean partitions.

5. STEP-WORD

The symbol we introduce to explain words in the Markoff tree is called the *step-word*

$$(A,B)^{a,b} = \prod_{s=1}^{a} AB^{e_s}, \quad e_s = [sb/a] - [(s-1)b/a]$$

where $n = [\xi]$ is the integral part of ξ (satisfying $n \le \xi < n + 1$). Here we assume $a > 0$, $b > 0$, and gcd $(a,b) = 1$. The further definition "by fiat" includes $a = 0$ $(b = 1)$,

$$(A,B)^{0,1} = B.$$

In any case, $(A,B)^{a,b}$ has coordinates $\{a, b\}$, $\left(\text{i.e., } \sum_{s=1}^{a} e_s = b\right)$.

Some of the simple cases are:

$$(A,B)^{1,0} = A, \quad (A,B)^{0,1} = B, \quad (A,B)^{1,1} = AB$$

$$(A,B)^{n,1} = A^nB, \quad (A,B)^{1,n} = AB^n, \quad (A,B)^{2m+1,2} = A^mBA^{m+1}B,$$

$$(A,B)^{2,2m+1} = AB^mAB^{m+1}, \quad (A,B)^{3,3m+2} = AB^mAB^{m+1}AB^{m+1}, \text{ etc.}$$

Note that the values of e_s (if more than one occurs) are chosen from *two* consecutive integers, $[b/a]$ and $[b/a] + 1$.

The symbol can be extended to an arbitrary integral pair (a,b) but this is not relevant to present work.

To see why the symbol is called a "step-word" let us note that the values of $e_1, \ldots, e_a$ are found by differencing the sequence $[bs/a]$ for $s = 0, 1, 2, \ldots, a$, in other words, by differencing the integral values of the *step-function* $y = [bx/a]$ lying just below the line $y = bx/a$ for $0 \le x \le a$.

6. NIELSEN PARTITION

We now construct a partition of step-words $w = (A,B)^{a,b}$ based on the euclidean partition of (a,b). (It is called a "Nielsen partition" for reasons explained in [1].) *The idea is that if*

$$(a,b) = (a',b') + (a'',b'')$$

is a euclidean partition, then the step-word has a (Nielsen) partition

$$(A,B)^{a,b} = (A,B)^{a',b'} \cdot (A,B)^{a'',b''}.$$

For example, since $(5,7) = (3,4) + (2,3)$, we obtain the partition:

$$ABABAB^2ABAB^2 = ABABAB^2 \cdot ABAB^2.$$

The justification is that the triangle bounded by the (integral) lattice points $(0,0)$, (a',b'), (a,b) has no lattice points in its interior and lies below the line $y = bx/a$ (since $ab' - ba' = -1$). Hence the step-function for $y = bx/a$ agrees with that of $y = b'x/a'$ for $0 \le x \le a'$ and agrees with that of the segment from (a',b') to (a,b) (of slope b''/a''), for

$$a' \le x \le a' + a'' = a.$$

Inductive property of Nielsen partituons. Let (w_0', w_0'') be a pair of words in the Markoff free. Assume that if $(w_0', w_0'')^+$ occurs, then $(w_0', w_0'') \sim (w', w'')$ with $w = w'w''$ a Nielsen partition (of step-words), and also assume that if $(w_0', w_0'')^-$ occurs, then $(w_0', w_0'') \sim (w', w'')$ with $w = w'w''$ a Nielsen partition (of step-words). *Then, the same property is hereditary to the next stage of the tree.*

The property is almost immediate, the only difficulty is in the order of the words. If we have $(w_0', w_0'')^+$ then if $(w_0', w_0'') \sim (w', w'')$ then $(w_0', w_0'', w_0'w_0'') \sim (w', w'', w'w'')$, so the property passes on to $(w_0', w_0'w_0'')^+$. On the other hand, $(w_0'', w_0', w_0'')^- \sim (w_0'', w_0''w_0')^+$, (using "$T$" = w_0''). Hence the property passes on to $(w_0'', w_0', w_0'w_0'')^-$ as well! The rest of the details are left to the reader.

7. MAIN THEOREM

If $w(A,B)$ is a word in the Markoff tree (with the coordinates $\{a, b\}$), then $a \geq 0$, $b \geq 0$, gcd $(a,b) = 1$, and

$$w(A,B) \sim (A,B)^{a,b}.$$

Conversely, for every pair (a,b) satisfying the above conditions, a representative $w(A,B)$ occurs in the Markoff tree.

The proof is a direct consequence of the inductive property of the euclidean partition and the Nielsen partition. Clearly, the first stage (A,B) gives a Nielsen partition $AB = A.B$!

A strange consequence of this result is that the same proof would hold if we used the step-word as $(B,A)^{b,a}$ instead. (Basically, this is a consequence of the relation $AB \sim BA$.) Thus, since the main theorem is now very clear on obtaining *both* $(A,B)^{a,b}$ and $(B,A)^{b,a}$, we have

$$(A,B)^{a,b} \sim (B,A)^{b,a}.$$

This is an elementary fact to verify but it is *not* trivial. For instance, if $(a,b) = (5,7)$, we have

$$ABABA.B^2ABAB^2 \sim B^2ABAB^2.ABABA$$

The dot indicates the point at which cyclic permutations would begin. The reader will find it amusing to explicitly write the T for which

$$(A,B)^{a,b}\, T = T(B,A)^{b,a}.$$

[It involves the congruence $bx \equiv -1 \pmod{a}$.]

8. MARKOFF TRIPLES

In conclusion, we shall indicate (without proofs) how some basic number-theoretic work of Markoff [2] leads to Markoff trees of words of a semigroup. The central device is the equation in positive integers defining a so-called *Markoff triple* (m_1, m_2, m_3)

$$m_1^2 + m_2^2 + m_3^2 = 3m_1m_2m_3, \quad (m_i > 0).$$

This so-called *Markoff equation* is discussed in [1] in terms of its connections with many branches of mathematics.

The important fact about the Markoff triple is that if $m_1^* = 3m_2m_3 - m_1$, $m_2^* = 3m_3m_1 - m_2$, $m_3^* = 3m_1m_2 - m_3$ then additional Markoff triples are verifiable as

$$(m_1^*, m_2, m_3),\ (m_1, m_2^*, m_3),\ (m_1, m_2, m_3^*).$$

The presence of three *neighbors* is exactly the property of the Markoff tree, one neighbor is the *ancestor* of (m_1, m_2, m_3) and two neighbors are *descendents*. The point is that all solutions can be obtained from (1,1,1) by neighbor formation, and if we consider only solutions which have *unequal* m_1, m_2, m_3, they can be obtained from (1,2,5). [Its neighbors are (29,2,5), (1,13,5) and (1, 2,1), which is excluded, see the tree below.]

The connection with the semigroup S_2 arises as follows: If $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$, then every word on the Markoff tree consists of a pair of matrices (w', w''). Then a general Markoff triple (of unequal m_i) is given (in some order) by

$$m_1 = \frac{1}{3} \text{ trace } w',\ m_2 = \frac{1}{3} \text{ trace } w'',\ m_3 = \frac{1}{3} \text{ trace } w'w''.$$

Since traces are equal for equivalent words, then, by the main theorem, the Markoff triple is given by step-words in a Nielsen partition $w'w'' = w$. Since the partition is unique, each triple is given by the coordinates $\{a, b\}$ of (say) w. The reader can verify that for $\{1, 1\}$, $(w', w'') = (A, B)$ and the triple (1,2,5) comes from 1/3 of the traces of A, B, and AB.

More generally, the Markoff tree of Section 3 leads to three solutions (rearranging the order so $m_1 < m_2 < m_3$):

$$\begin{tikzcd}[column sep=tiny]
 & & & \{1,\ 1\}(1,2,5) \arrow[dll] \arrow[drr] & & & \\
 & \{2,\ 1\}(1,5,13) \arrow[dl] \arrow[dr] & & & & \{1,\ 2\}(2,5,29) \arrow[dl] \arrow[dr] & \\
\{3,\ 1\}(1,13,34) & & \{3,\ 2\}(5,13,194) & & \{1,\ 3\}(2,29,169) & & \{2,\ 3\}(5,29,433)
\end{tikzcd}$$

A result which is still a troublesome conjecture (see [3]), is that there exists a unique nonnegative pair (a,b) for which the matrix

$$M = \left(\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}\right)^{a,b}$$

has a given trace. Thus, m_3 (= 1/3 trace M) determines m_1, m_2 completely (if we keep $m_1 < m_2 < m_3$ as before).

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