ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS *to* Professor A. P. Hillman, 709 Solano Dr., S. E., Albuquerque, New Mexico 87108. *Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within 4 months of the publication date.*

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-400 *Proposed by Herta T. Freitag, Roanoke, VA*

Let T_n be the nth triangular number $n(n + 1)/2$. For which positive integers n is $T_1^2 + T_2^2 + \cdots + T_n^2$ an integral multiple of T_n?

B-401 *Proposed by Gary L. Mullen, Pennsylvania State University, Sharon, PA*

Show that $\lim\limits_{n\to\infty} [(n!)^{2n}/(n^2)!] = 0$.

B-402 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Show that $(L_nL_{n+3},\ 2L_{n+1}L_{n+2},\ 5F_{2n+3})$ is a Pythagorean triple.

B-403 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Let $m = 5^n$. Show that $L_{2m} \equiv -2 \pmod{5m^2}$.

B-404 *Proposed by Phil Mana, Albuquerque, NM*

Let x be a positive irrational number. Let a, b, c, and d be positive integers with $a/b < x < c/d$. If $a/b < r < x$, with r rational, implies that the denominator of r exceeds b, we call a/b a good lower approximation (GLA) for x. If $x < r < c/d$, with r rational, implies that the denominator of r exceeds d, c/d is a good upper approximation (GUA) for x. Find all the GLAs and all the GUAs for $(1 + \sqrt{5})/2$.

B-405 *Proposed by Phil Mana, Albuquerque, NM*

Prove that for every positive irrational x, the GLAs and GUAs for x (as defined in B-404) can be put together to form one sequence $\{p_n/q_n\}$ with

$$p_{n+1}q_n - p_nq_{n+1} = \pm 1 \quad \text{for all } n.$$

SOLUTIONS

Complementary Primes

B-376 *Proposed by Frank Kocher and Gary L. Mullen, Pennsylvania State University, University Park and Sharon, PA*

Find all integers $n > 3$ such that $n - p$ is an odd prime for all odd primes p less than n.

Solution by Paul S. Bruckman, Concord, CA

Let n be a solution to the problem, and p any odd prime less than n. Since p and $n - p$ are odd, clearly n must be even. Hence, $n \equiv 0, 2, 4 \pmod 6$. Since $4 - 3 = 6 - 5 = 8 - 7 = 1$ and 1 is not a prime, it follows that $n \neq 4$, $n \neq 6$, $n \neq 8$. Hence, $n \geq 10$.

If $n \equiv 0 \pmod 6$, then $n - 3 \equiv 3 \pmod 6$, which shows that $n - 3$ is composite and ≥ 9. Likewise, if $n \equiv 2 \pmod 6$, then $n - 5 \equiv 3 \pmod 6$, which shows that $n - 5$ is composite and ≥ 9. Finally, if $n \equiv 4 \pmod 6$, then $n - 7 \equiv 3 \pmod 6$, which is composite, *unless* $n = 10$, in which case $n - 7 = 3$, a prime. Hence, $n = 10$ is the only possible solution. Since $10 - 3 = 7$, $10 - 5 = 5$, $10 - 7 = 3$, which are all primes, $n = 10$ is indeed the only solution to the problem.

Also solved by Heiko Harborth (W. Germany), Charles Joscelyne, Graham Lord, J. M. Metzger, Bob Prielipp, E. Schmutz & M. Wachtel (Switzerland), Sahib Singh, Rolf Sonntag (W. Germany), Charles W. Trigg, Gregory Wulczyn, and the proposer.

Counting Lattice Points

B-377 *Proposed by Paul S. Bruckman, Concord, CA*

For all real numbers $a \geq 1$ and $b \geq 1$, prove that

$$\sum_{k=1}^{[a]} \left[b\sqrt{1 - (k/a)^2} \right] = \sum_{k=1}^{[b]} \left[a\sqrt{1 - (k/b)^2} \right],$$

where $[x]$ is the greatest integer in x.

Solution by J. M. Metzger, University of North Dakota, Grand Forks, ND

Each sum counts the number of lattice points in the first quadrant of

$$\frac{x^2}{a^2} + \frac{x^2}{b^2} = 1,$$

the first along the vertical lines, $x = 1$, $x = 2$, $\ldots$, $x = [a]$, the second along the horizontal lines, $y = 1$, $y = 2$, $\ldots$, $y = [b]$. The two counts must agree.

Also solved by Bob Prielipp, Sahib Singh, and the proposer.

Congruence Mod 3

B-378 *Proposed by George Berzsenyi, Laram University, Beaumont, TX*

Prove that $F_{3n+1} + 4^n F_{n+3} \equiv 0 \pmod 3$ for $n = 0, 1, 2, \ldots$.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oskosh, WI

We shall establish that $F_{3n+1} + F_{n+3} \equiv 0 \pmod 3$ for $n = 0, 1, 2, \ldots,$ which is equivalent to the stated result because $4^n \equiv 1 \pmod 3$ for each nonnegative integer n. Clearly the desired result holds when $n = 0$ and when $n = 1$. Assume that $F_{3k+1} + F_{k+3} \equiv 0 \pmod 3$ and $F_{3k+4} + F_{k+4} \equiv 0 \pmod 3$, where k is an arbitrary nonnegative integer. Then, by addition,

$$F_{3k+1} + F_{3k+4} + F_{k+5} \equiv 0 \pmod 3.$$

But

$$6F_{3k+2} + 4F_{3k+1} + F_{3k+4} = F_{3k+7}$$

so

$$F_{3k+1} + F_{3k+4} \equiv F_{3k+7} \pmod 3.$$

Hence

$$F_{3k+7} + F_{k+5} \equiv 0 \pmod 3$$

and our proof is complete by mathematical induction.

Also solved by Paul S. Bruckman, Herta T. Freitag, Graham Lord, Sahib Singh, Gregory Wulczyn, and the proposer.

Congruence Mod 5

B-379 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that $F_{2n} \equiv n(-1)^{n+1} \pmod 5$ for all nonnegative integers n.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, WI

Clearly the desired result holds when $n = 0$ and when $n = 1$. Assume that $F_{2k} \equiv k(-1)^{k+1} \pmod 5$ and $F_{2k+2} \equiv (k+1)(-1)^{k+2} \pmod 5$, where k is an arbitrary nonnegative integer. Then, since

$$\begin{aligned} F_{2k+4} &= 3F_{2k+2} - F_{2k}, \\ F_{2k+4} &\equiv (3k+3)(-1)^{k+2} - k(-1)^{k+1} \pmod 5 \\ &\equiv (-1)^{k+2}(4k+3) \pmod 5 \\ &\equiv (k+2)(-1)^{k+3} \pmod 5. \end{aligned}$$

Our solution is now complete by mathematical induction.

Also solved by Paul S. Bruckman, Charles Joscelyne, Graham Lord, Sahib Singh, Gregory Wulczyn, and the proposer.

Binomial Convolution

B-380 *Proposed by Dan Zwillinger, Cambridge, MA*

Let a, b, and c be nonnegative integers. Prove that

$$\sum_{k=1}^{n} \binom{k+a-1}{a}\binom{n-k+b-c}{b} = \binom{n+a+b-c}{a+b+1}.$$

Here $\binom{m}{r} = 0$ if $m < r$.

Solution by Phil Mana, Albuquerque, NM

For every nonnegative integer d, the Maclaurin series for $(1 - x)^{-d-1}$ is

$$\sum_{n=0}^{\infty} \binom{n+d}{d} x^n.$$

Then

$$(1 - x)^{-a-1}(1 - x)^{-b-1} = (1 - x)^{-a-b-2},$$

$$\sum_{i=0}^{\infty} \binom{i+a}{a} x^i \cdot \sum_{j=0}^{\infty} \binom{j+b}{b} x^j = \sum_{n=0}^{\infty} \binom{n+a+b+1}{a+b+1} x^n.$$

Equating coefficients of x^{n-c-1} on both sides, one has

$$\sum_{k=1}^{n-c} \binom{k-1+a}{a}\binom{n-c-k+b}{b} = \binom{n-c+a+b}{a+b+1}$$

The upper limit $n - c$ for the sum here can be replaced by n, since any terms for $n - c < k \le n$ will vanish using the convention that $\binom{m}{r} = 0$ for $m < r$. This gives the desired result.

Also solved by Paul S. Bruckman, Bob Prielipp & N. J. Kuenzi, A. G. Shannon, and the proposer.

Generating Function

B-381 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Let $a_{2n} = F_{n+1}^2$ and $a_{2n+1} = F_{n+1}F_{n+2}$. Find the rational function that has

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

as its Maclaurin series.

Solution by Sahib Singh, Clarion State College, Clarion, PA

By the result $\sum_{i=1}^{n} F_i^2 = F_nF_{n+1}$, we get the Mclaurin series as:

$$F_1^2 + F_1^2x(1 + x^2 + x^4 + \cdots) + F_2^2X_2^2 + F^2X^3(1 + x^2 + x^4 + \cdots) + \cdots$$

$$= F_1^2\left(1 + \frac{x}{1 - x^2}\right) + F_2^2X^2\left(1 + \frac{x}{1 - x^2}\right) + F_3^2X^4\left(1 + \frac{x}{1 - x^2}\right) + \cdots$$

$$= \frac{1 + x - x^2}{1 - x^2}\left[F_1^2 + F_2^2X^2 + F_3^2X^4 + F_4^2X^6 + \cdots\right].$$

Using $F_n^2 = \left(\frac{a^n - b^n}{a - b}\right)^2$, the above becomes

$$\left(\frac{1+x-x^2}{1-x^2}\right)\cdot\frac{1}{(a-b)^2}\left[(a^2+a^4x^2+a^6x^4+\cdots)\right.$$
$$\left.+(b^2+b^4x^2+b^6x^4+\cdots)-2ab(1+abx^2+a^2b^2x^4+\cdots)\right]$$
$$=\left(\frac{1+x-x^2}{1-x^2}\right)\cdot\frac{1}{(a-b)^2}\left[\frac{a^2}{1-a^2x^2}+\frac{b^2}{1-b^2x^2}-\frac{2ab}{1-abx^2}\right],$$

which simplifies to

$$\left(\frac{1+x-x^2}{1-x^2}\right)\left(\frac{(1-x^2)}{(1+x^2)(1-3x^2+x^4)}\right)=\frac{1+x-x^2}{(1+x^2)(1-3x^2+x^4)}.$$

Also solved by Paul S. Bruckman, R. Garfield, John W. Vogel, and the proposer.

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ERRATA

The following errors have been noted:

Volume 16, No. 5 (October 1978), p. 407 [J. A. H. Hunter's "Congruent Primes of Form $(8r+1)$"]. The equations presented in the second line of the article should read

$$X^2 - eY^2 = Z^2, \text{ and } X^2 + eY^2 = W^2.$$

Volume 17, No. 1 (February 1979), p. 84 (A. P. Hillman & V. E. Hoggatt, Jr.'s "Nearly Linear Functions"). Equation (1) should read

$$C'\cdot H - C\cdot H = \sum_{i=1}^{k}(c_i' - c_i)h_i \geq h_k - \sum_{i=1}^{k-1} c_i h_i. \tag{1}$$

The second line of the proof of Lemma 7 should read

The hypothesis $E\cdot E' = 0$ implies . . . .

In the proof of Theorem 1, Equation (10) should read

$$b_j(m) = C^*_{m-1}\cdot H_j - C_{m-1}\cdot H_j. \tag{10}$$

(Kindness of Margaret Owens)