MORE IN THE THEORY OF SEQUENCES

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INTRODUCTION

Cauchy gave a necessary condition for the convergence of an infinite series,

$$\sum_{k=1}^{\infty} a(k);$$

namely, that the sequence $(a(n))$ converges to zero as n tends to infinity.

Olivier proved a variation of this theorem, which has, in a sense, generated more interest: Let $(a(n))$ be a monotonic nonincreasing sequence of positive numbers, tending to zero, such that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a(k)$$

exists, then $\lim_{n\to\infty} n \cdot a(n) = 0$.

For one thing, Olivier's theorem allows for extensions in several directions [4]. Niven and Zuckerman, for instance, have proved the following theorem [5]:

Theorem 1: Let $(a(n))$ be a monotonic nonincreasing sequence of positive numbers. Then

$$\lim_{n\to\infty} \sum_{k=n+1}^{[\lambda n]} a(k) \tag{1}$$

exists for each $\lambda > 1$, if and only if $\lim_{n\to\infty} n \cdot a(n)$ exists.

Clearly, Niven and Zuckerman's condition for the convergence of

$$(n \cdot a(n))$$

is weaker than that of Olivier. On the other hand, they have given a necessary and sufficient condition for the convergence of

$$\left(\sum_{k=n+1}^{[\lambda n]} a(k)\right).$$

In this paper, Olivier's theorem will be extended further in this same direction. We consider a sequence of positive numbers $(\emptyset(n))$ (as yet unspecified) and a monotonic nonincreasing sequence of positive numbers $(a(n))$, such that

$$\lim_{n\to\infty} \frac{1}{\emptyset(n)} \sum_{k=n+1}^{[\lambda n]} a(k)$$

exists for every $\lambda > 1$. We will show that $\lim_{n\to\infty} \dfrac{n \cdot a(n)}{\emptyset(n)}$ exists.

When $\phi(n) = 1$, $n = 1, 2, 3, \ldots,$ the problem reduces to the case considered by Niven and Zuckerman. But more generally, as we will prove, $(\phi(n))$ can be any regularly varying sequence, i.e., any sequence of positive numbers which satisfies

$$(2) \qquad \lim_{n\to\infty} \frac{\phi([\lambda n])}{\phi(n)} = \psi(\lambda) \text{ for every } \lambda > 0,$$

where $\psi(\lambda) = \lambda^{\rho}$, where the index ρ is real.

We summarize this result in Theorem 2.

Theorem 2: Let $(\phi(n))$ be a regularly varying sequence and let $(a(n))$ be a monotonic nonincreasing sequence of positive numbers. Then

$$(3) \qquad \lim_{n\to\infty} \frac{1}{\phi(n)} \sum_{k=n+1}^{[\lambda n]} a(k) = H(\lambda)$$

exists for each $\lambda > 1$, if and only if $\lim\limits_{n\to\infty} \dfrac{n \cdot a(n)}{\phi(n)}$ exists.

Proof: Let

$$H(\lambda) = \lim_{n\to\infty} H_n(\lambda) = \lim_{n\to\infty} \frac{1}{\phi(n)} \sum_{k=n+1}^{[\lambda n]} a(k).$$

For each integer $m > \lambda$, let $n = [m/\lambda]$ in $H_n(\lambda)$ and let $r = m - [n\lambda]$. Since $0 = m - m/\lambda \cdot \lambda \leq m - n \cdot \lambda$, we have $m \geq n\lambda = [n\lambda]$. Also,

$$0 \leq r = m - [n\lambda] < m - (n\lambda - 1) < m - (m/\lambda - 1) = \lambda + 1.$$

Since

$$H_n(\lambda) \geq \frac{([n\lambda] - n) \cdot a([n\lambda])}{\phi(n)} \geq \frac{([n\lambda] - n) \cdot a(m)}{\phi(n)}$$

and

$$\frac{[n\lambda] + r}{[n\lambda] - n} \leq \frac{n\lambda + \lambda + 1}{n\lambda - 1 - n} \leq \frac{m + \lambda + 1}{(m/\lambda - 1)\lambda - 1 - n} \leq \frac{m + \lambda + 1}{m - \lambda - 1 - m/\lambda},$$

we have

$$\frac{m \cdot a(m)}{\phi(n)} = \frac{m \cdot a(m)}{\phi(n)} \cdot \frac{\phi(n)}{\phi(m)} \leq \frac{[n\lambda] + r}{[n\lambda] - n} \cdot H_n(\lambda) \cdot \frac{\phi[m/\lambda]}{\phi(m)}$$

$$\leq \frac{m + \lambda + 1}{m - \lambda - 1 - m/\lambda} \cdot H_n(\lambda) \cdot \frac{\phi([m/\lambda])}{\phi(m)}.$$

Hence, by (2),

$$(4) \qquad \lim_{n\to\infty} \sup \frac{m \cdot a(m)}{\phi(m)} \leq \frac{\lambda}{\lambda - 1} \cdot H(\lambda) \cdot (1/\lambda)^{\rho}.$$

We assert that

$$\lim_{n\to\infty} \frac{A([\lambda\mu n]) - A([\mu n])}{\phi([\mu n])} = H(\lambda),$$

where $\lambda > 1$, $\mu > 0$, and

$$A([\lambda n]) = \sum_{k=1}^{[\lambda n]} a(k).$$

It is sufficient to show

$$\lim_{n\to\infty} \frac{1}{\phi([\mu n])} \sum_{k=(\lambda[\mu n])+1}^{[\lambda\mu n]} a(k) = 0,$$

since

$$\frac{A([\lambda\mu n]) - A([\mu n])}{\phi([\mu n])} = H_{[\mu n]}(\lambda) + \frac{1}{\phi([\mu n])} \sum_{k=(\lambda[\mu n])+1}^{[\lambda\mu n]} a(k).$$

Clearly, by (2) and (4),

$$\overline{\lim_{n\to\infty}} \frac{1}{\phi([\mu n])} \sum_{k=(\lambda[\mu n])+1}^{[\lambda\mu n]} a(k)$$

$$= \overline{\lim_{n\to\infty}} \frac{\phi(\lambda[\mu n])}{\phi([\mu n])} \cdot \frac{([\lambda] + 2)}{[\lambda[\mu n]]} \cdot \frac{[\lambda[\mu n]]a([\lambda[\mu n]])}{\phi([\lambda[\mu n]])} = 0,$$

so our assertion is proved.

Therefore, we have

$$H(\lambda\mu) = H(\lambda)\mu^{\rho} + H(\mu), \tag{5}$$

since

$$H_n(\lambda\mu) = \frac{A([\lambda\mu n]) - A([\mu n])}{\phi([\mu n])} \cdot \frac{\phi([\mu n])}{\phi(n)} + \frac{A([\mu n]) - A(n)}{\phi(n)}.$$

Interchanging μ with λ in (5) and manipulating the equations simultaneously, we have, if $\rho \neq 0$, $H(\mu)/\mu^{\rho} - 1 = H(\lambda)/\lambda^{\rho} - 1 = A$, A a constant, which implies

$$H'(1) = \lim_{\lambda\to 1^+} \frac{H(\lambda)}{\lambda - 1} = \lim_{\lambda\to 1^+} \frac{H(\lambda)}{\lambda^{\rho} - 1} \cdot \lim_{\lambda\to 1^+} \frac{\lambda^{\rho} - 1}{\lambda - 1} = A \cdot \rho,$$

or

$$H(\lambda) = \frac{H'(1)}{\rho}(\lambda^{\rho} - 1). \tag{6}$$

If $\rho = 0$, then $H(\lambda\mu) = H(\lambda) + H(\mu)$. Since $H(\cdot)$ is monotonic increasing, $H(\cdot)$ has a point of continuity and it is not hard to show $H(\cdot)$ is continuous on $[1,\infty]$. Hence $H(\cdot)$ is of the form

$$H(\lambda) = H'(1) \log \lambda. \tag{7}$$

Since

$$H(\lambda) \leq \lim_{n\to\infty} \frac{n \cdot a(n)}{\phi(n)} \cdot \frac{([\lambda n] - n)}{n} = (\lambda - 1) \lim_{n\to\infty} \frac{n \cdot a(n)}{\phi(n)},$$

we have

$$H'(1) = \lim_{\lambda\to 1} \frac{H(\lambda)}{\lambda - 1} = \lim_{n\to\infty} \frac{n \cdot a(n)}{\phi(n)}.$$

On the other hand, as a consequence of (4), we have

$$\frac{H(\lambda)}{\lambda^{\rho}} = \lim_{n\to\infty} \frac{A(n) - A([n/\lambda])}{\emptyset(n)} \geq \lim_{n\to\infty} \sup \frac{(n - [n/\lambda])}{n} \cdot \frac{n \cdot a(n)}{\emptyset(n)}.$$

Therefore, from (6) and (7),

$$H'(1) = \lim_{\lambda\to 1^+} \sup \frac{H(\lambda)}{\lambda^{\rho}(1 - 1/\lambda)} = \lim_{n\to\infty} \sup \frac{n \cdot a(n)}{\emptyset(n)}.$$

Hence,

$$\lim_{n\to\infty} \frac{n \cdot a(n)}{\emptyset(n)} = H'(1).$$

We now prove the converse.

Definition: Let $f(x)$ be a real valued, measurable function which satisties

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^{\rho}$$

for every $\lambda > 0$. Then $f(x)$ is a regularly varying function of index ρ.

Every regularly varying function $f(x)$ of index ρ can be written as

$$f(x) = \lambda^{\rho} L(x) \tag{8}$$

where $L(x)$ is regularly varying of index 0 (slowly varying). (See [2].)

Lemma 1: Let $(\emptyset(n))$ be a regularly varying sequence of index ρ, then the function $f(x)$ defined by

$$f(x) = \emptyset([x])$$

is a regularly varying function of index ρ.

Lemma 2: If $L(x)$ is a slowly varying function, then for every $[a,b]$, $0 < a < b < \infty$, the relation

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1$$

holds uniformly with respect to $x \in [a,b]$.

Lemma 2, known as the Uniform Convergence Theorem for slowly varying functions, has been proved by several persons. A nice proof is given in [1] by Bojanic and Seneta. Lemma 1 is proved by the author in [3].

By hypothesis,

$$\lim_{k\to\infty} \frac{k \cdot a(k)}{\emptyset(k)} = C.$$

Also, by (8), $\emptyset(k)$ can be written as

$$\emptyset(k) = k^{\rho} L(k),$$

where $L(k)$ is slowly varying. Therefore, $(a(k))$ can be written as

$$a(k) = C(k) k^{\rho - 1} L(k),$$

where $\lim_{k\to\infty} C(k) = C$.

Consequently, for n sufficiently large,

$$\frac{(C-\varepsilon)}{n^{\rho}} \min_{n \le k \le [\lambda n]} \frac{L(k)}{L(n)} \sum_{k=n+1}^{[\lambda n]} k^{\rho-1} \le \frac{1}{\emptyset(n)} \sum_{k=n+1}^{[\lambda n]} a(k)$$
$$\le \frac{(C-\varepsilon)}{n^{\rho}} \max_{n \le k \le [\lambda n]} \frac{L(k)}{L(n)} \sum_{k=n+1}^{[\lambda n]} k^{\rho-1},$$

where $\varepsilon > 0$.

Clearly,

$$\min_{n \le k \le [\lambda n]} \frac{L(k)}{L(n)} = \min_{1 \le k' \le \lambda} \frac{L(k'n)}{L(n)}$$

and

$$\max_{n \le k \le [\lambda n]} \frac{L(k)}{L(n)} = \max_{1 \le k' \le \lambda} \frac{L(k'n)}{L(n)}.$$

By Lemmas 1 and 2, we have

$$\lim_{n\to\infty} \min_{1 \le k' \le \lambda} \frac{L(k'n)}{L(n)} = 1 = \overline{\lim_{n\to\infty}} \max_{1 \le k' \le \lambda} \frac{L(k'n)}{L(n)}.$$

Therefore,

$$\frac{(C-\varepsilon)}{n^{\rho}} \sum_{k=n+1}^{[\lambda n]} k^{\rho-1} = \lim_{n\to\infty} \frac{1}{\emptyset(n)} \sum_{k=n+1}^{[\lambda n]} a(k) = \overline{\lim_{n\to\infty}} \frac{1}{\emptyset(n)} \sum_{k=n+1}^{[\lambda n]} a(k) = \frac{(C+\varepsilon)}{n^{\rho}} \sum_{k=n+1}^{[\lambda n]} k^{\rho-1}.$$

On the other hand,

$$\sum_{k=n+1}^{[\lambda n]} k^{\rho-1} \simeq \begin{cases} \dfrac{(\lambda^{\rho}-1)n^{\rho}}{\rho} & \text{if } \rho \ne 0 \\ \log \lambda & \text{if } \rho = 0 \end{cases}$$

Hence, letting $\varepsilon \to 0$, we have

$$\lim_{n\to\infty} \frac{1}{\emptyset(n)} \sum_{k=n+1}^{[\lambda n]} a(k) = \begin{cases} \dfrac{C(\lambda^{\rho}-1)}{\rho} & \text{if } \rho \ne 0 \\ C \log \lambda & \text{if } \rho = 0 \end{cases}$$

and the converse is proved.

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