(11) "p" is a prime if and only if it appears exactly $(p - 1)$ times in line $(p - 1)$.

(12) $s(n,r)$ will appear again at locations $s(n + k,\ 2^k(r - 1) + 1)$ for $k = 1, 2, 3, \ldots$.

(13) If the sequence r_1, r_2 occurs in row n, $r_1 > r_2$, the smallest element in row $n + k$ positioned between r_1 and r_2 is
$s(n + k,\ 2^k r) = r_1 + kr_2$.

(14) In any row, there are two equal terms greater than all others in the row.

(15) For Fibonacci followers:
$s(n,r) = F_{n+1}$, for $r = (2^{n-1} + 2 + \{1 + (-1)^n\})/3 - 1$,
and it is the largest element in the row.
(See [3], p. 65; notation changed to standard form.)

Not all of the discovered results are considered here, since there are remote connections to so many areas of number theory.

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SUMS OF PRODUCTS: AN EXTENSION

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The purpose of this note is to extend the results of Berzsenyi [1] and Zeilberger [3] on sums of products by using the generalized sequence

$$\{W_n(a,b;p,q)\}$$

described by the author in [2], the notation of which will be assumed.

Equation (4.18) of [2, p. 173] tells us that

$$W_{n-r}W_{n+r+t} - W_nW_{n+t} = eq^{n-r}U_{r-1}U_{r+t-1}. \tag{1}$$

Putting $n - r = k$ and summing appropriately, we obtain

$$\sum_{k=0}^{n} W_kW_{k+2r+t} = \sum_{k=0}^{n} W_{k+r}W_{k+r+t} + eU_{r-1}U_{r+t-1}\sum_{k=0}^{n} q^k. \tag{2}$$

Values $t = 1$, $t = 0$ give, respectively,

$$\sum_{k=0}^{n} W_kW_{k+2r+1} = \sum_{k=0}^{n} W_{k+r}W_{k+r+1} + eU_{r-1}U_r\sum_{k=0}^{n} q^k, \tag{3}$$

and

$$\sum_{k=0}^{n} W_k W_{k+2r} = \sum_{k=0}^{n} W_{k+r}^2 + eU_{r-1}^2 \sum_{k=0}^{n} q^k. \tag{4}$$

If $q = -1$, then

$$\sum_{k=0}^{n} q^k = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{5}$$

Using the Binet form for W_n and U_n, we find after calculation that (3) and (4), under the restrictions (5), become, respectively,

$$\sum_{k=0}^{n} W_k W_{k+2r+1} = \begin{cases} \frac{1}{p}(W_{r+n+1}^2 - W_{r+1}^2) - W_0 W_{2r+1} & \text{if } n \text{ is even} \\ \frac{1}{p}(W_{r+n+1}^2 - W_r^2) & \text{if } n \text{ is odd,} \end{cases} \tag{6}$$

and

$$\sum_{k=0}^{n} W_k W_{k+2r} = \begin{cases} \frac{1}{p}(W_{r+n} W_{r+n+1} - W_r W_{r+1}) + W_0 W_{2r} & \text{if } n \text{ is even} \\ \frac{1}{p}(W_{r+n} W_{r+n+1} - W_{r-1} W_r) & \text{if } n \text{ is odd.} \end{cases} \tag{7}$$

When $p = 1$, so that $W_n = H_n$ (and $U_n = F_n$), (6) and (7) reduce to the four formulas given by Berzsenyi [1]. That is, Berzsenyi's four formulas are special cases of (1), i.e., of equation (4.18) of [2].

Zeilberger's theorem [3] then generalizes as follows:

Theorem: If $\{Z_n\}$ and $\{W_n\}$ are two generalized Fibonacci sequences, in which $q = -1$, then

$$\sum_{i,j=0}^{n} a_{ij} Z_i W_j = 0$$

if and only if

$$P(z,\omega) = \sum_{i,j=0}^{n} a_{ij} z^i \omega^j$$

vanishes on $\{(\alpha,\alpha), (\alpha,\beta), (\beta,\alpha), (\beta,\beta)\}$ where α,β are the roots of

$$x^2 - px - 1 = 0.$$

Zeilberger's example [3[now refers to

$$\sum_{k=0}^{n} Z_k W_{k+2r+1} = \frac{1}{p}(Z_{r+n+1} W_{r+n+1} - Z_{r+1} W_{r+1}) + Z_0 W_{2r+1}. \tag{8}$$

(In both [1] and [3], m is used instead of our r.)

Verification of the above results involves routine calculation. Difficulties arise when $q \neq -1$.

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A CONJECTURE IN GAME THEORY

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We consider a team composed of n players, with each member playing the same r games, $G_1, G_2, \ldots, G_r$. We assume that each game G_j has two possible outcomes, success and failure, and that the probability of success in game G_j is equal to p_j for each player. We let X_{ij} be equal to one (1) if player i has a success in game j and let X_{ij} be equal to zero (0) if player i has a failure in game j. We assume throughout this paper that the random variables X_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, r$ are independent.

Let S_{jn} denote the total number of successes in the jth game. We define the point-value of a team to be

$$\Psi_n = \min_{1 \le j \le r} S_{jn}.$$

This means that the point-value of a team is equal to the minimum number of successes in any particular game. Clearly,

$$P\{S_{jn} = m\} = \binom{n}{m} p_j^m (1 - p_j)^{n-m}, \quad m = 0, 1, 2, \ldots, n,$$

and

$$
\begin{aligned}
(1) \qquad E[\Psi_n] &= \sum_{k=0}^{n} kP\{\Psi_n = k\} = \sum_{k=0}^{n-1} P\{\Psi_n > k\} \\
&= \sum_{k=0}^{n-1} P\{S_{1n} > k, S_{2n} > k, \ldots, S_{rn} > k\} \\
&= \sum_{k=0}^{n-1} \prod_{j=1}^{r} P\{S_{jn} > k\} \\
&= \sum_{k=0}^{n-1} \prod_{j=1}^{r} \sum_{m=k+1}^{n} \binom{n}{m} p_j^m (1 - p_j)^{n-m}.
\end{aligned}
$$

It follows from the definition of Ψ_n that the expected point-value for a team is an increasing function of n, i.e.,

$$E[\Psi_n] \le E[\Psi_{n+1}], \quad n = 1, 2, 3, \ldots .$$

Since a team can add players in order to increase its expected point-value, it seems reasonable to define the score to be the expected point-value per player. Namely, we denote the score by

$$W_n = \frac{1}{n} E[\Psi_n].$$