# RESTRICTED COMPOSITIONS II

L. CARLITZ
*Duke University, Durham, NC 27706*

## 1. INTRODUCTION

In [1], the writer considered the number of compositions

$$n = a_1 + a_2 + \cdots + a_k, \tag{1.1}$$

where the $a_i$ are either nonnegative or strictly positive and in addition

$$a_i \neq a_{i+1} \quad (i = 1, 2, \ldots, k - 1). \tag{1.2}$$

In the present paper, we consider the number of compositions (1.1) in nonnegative $a_j$ that satisfy

$$a_i \not\equiv a_{i+1} \pmod{m} \quad (i = 1, 2, \ldots, k - 1), \tag{1.3}$$

where $m$ is a fixed positive integer.

For $n \geq 0$, $k \geq 1$, let $f_m(n,k)$ denote the number of solutions of (1.1) and (1.3) and let

$$f_m(n) = \sum_{k=1}^{\infty} f_m(n,k) \tag{1.4}$$

denote the corresponding enumerant when the number of parts in (1.1) is unrestricted. Also, for $0 \leq j < m$, let $f_{m,j}(n,k)$ denote the number of solutions of (1.1) and (1.3) with $a_1 \equiv j \pmod{m}$.

For $m = 2$ explicit results are obtained, in particular,

$$f_{2,i}(n,k) = \binom{k + s - 1}{s} \quad (i = 0, 1), \tag{1.5}$$

where

$$s = \frac{1}{2}\left(n - \frac{1}{2}(k + i)\right) \tag{1.6}$$

and $[x]$ is the greatest integer $\leq x$.

For arbitrary $m \geq 1$, we show in particular that

$$\sum_{n,k=0}^{\infty} f_m(n,k)x^n y^k = \frac{P_m(z)}{Q_m(z)} \quad \left(z = \frac{y}{1 - x^m}\right), \tag{1.7}$$

where

$$P_m(z) = \prod_{j=0}^{m-1} (1 + x^j z)$$

and

$$Q_m(z) = P_m(z) - zP_m'(z).$$

For additional results, see Section 4 below.

## SECTION 2

In order to evaluate $f_m(n,k)$, we define the following functions. Let $f_{m,j}(n,k)$, where $n \geq 0$, $k \geq 1$, $0 \leq j < m$, denote the number of solutions in nonnegative integers of

$$n = a_1 + a_2 + \cdots + a_k, \tag{2.1}$$

where

$$(2.2)\qquad a_i \not\equiv a_{i+1} \pmod m \qquad (i = 1, 2, \ldots, k-1)$$

and

$$(2.3)\qquad a_1 \equiv j \pmod m.$$

Also let $f_{m,j}(n,k,a)$ denote the number of solutions of (2.1), (2,2), (2.3), with $a_1 = a$. Thus $f_{m,j}(n,k,a) = 0$ if $a \not\equiv j \pmod m$.

It is convenient to extend the above definitions to include the case $k = 0$. We put

$$(2.4)\qquad f_m(n,0) = \delta_{n0},$$

where $\delta_{ij}$ is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j). \end{cases}$$

We also define

$$(2.5)\qquad f_{m,j}(n,0) = \delta_{j0}\delta_{n0}$$

and

$$(2.6)\qquad f_{m,j}(n,0,a) = \delta_{j0}\delta_{n0}\delta_{a0},$$

that is, $f_{m,j}(n,0) = 0$ unless $n = j = 0$ and $f_{m,j}(n,0,a) = 0$ unless $n = j = a = 0$.

It follows from the definitions that

$$(2.7)\qquad f_m(n,k) = \sum_{j=0}^{m-1} f_{m,j}(n,k) = \sum_{j=0}^{m-1}\sum_{a=0}^{n} f_{m,j}(n,k,a) \qquad (n \geq 0,\ k \geq 0).$$

Moreover, we have the recurrence

$$f_{m,j}(n,k,a) = \sum_{\substack{i=0 \\ i \neq j}}^{m-1}\sum_{b=0}^{n-a} f_{m,i}(n-a,k-1,b) \qquad [k > 0,\ a \equiv j \pmod m],$$

which reduces to

$$(2.8)\qquad f_{m,j}(n,k,a) = \sum_{\substack{i=0 \\ i \neq j}}^{m-1} f_{m,i}(n-a,k-1) \quad [k > 0,\ a \equiv j \pmod m].$$

Corresponding to the various enumerants we define a number of generating functions:

$$F_{m,j}(x,y) = \sum_{n,k=0}^{\infty} f_{m,j}(n,k)x^n y^k$$

$$F_m(x,y) = \sum_{n,k=0}^{\infty} f_m(n,k)x^n y^k$$

$$F_{m,j}(x,y,a) = \sum_{n,k=0}^{\infty} f_{m,j}(n,k,a)x^n y^k.$$

## SECTION 3

We first discuss the case $m = 2$. The recurrence (2.8) reduces to

$$\text{(3.1)}\qquad \begin{cases} f_{2,0}(n,k,2a) = f_{2,1}(n-2a,k-1) \quad (k > 1), \\ \quad f_0(n,1,2a) = \delta_{n,2a} \\ f_{2,1}(n,k,2a+1) = f_{2,0}(n-2a-1,k-1) \quad (k \geq 1). \end{cases}$$

Hence,

$$\begin{cases} F_{2,0}(x,y,2a) = \delta_{a,0} + x^{2a}y + x^{2a}yF_{2,1}(x,y) \\ F_{2,1}(x,y,2a+1) = x^{2a+1}yF_{2,0}(x,y). \end{cases}$$

Summing over $a$, we get

$$\begin{cases} F_{2,0}(x,y) = 1 + \dfrac{y}{1-x^2} + \dfrac{y}{1-x^2}F_{2,1}(x,y) \\ F_{2,1}(x,y) = \dfrac{xy}{1-x^2}F_{2,0}(x,y). \end{cases}$$

It follows that

$$\text{(3.2)}\qquad F_{2,0}(x,y) = \frac{1 + \dfrac{y}{1-x^2}}{1 - \dfrac{xy^2}{(1-x^2)^2}}, \quad F_{2,1}(x,y) = \frac{\dfrac{xy}{1-x^2}\left(1 + \dfrac{y}{1-x^2}\right)}{1 - \dfrac{xy^2}{(1-x^2)^2}},$$

so that

$$\text{(3.3)}\qquad F_2(x,y) = F_{2,0}(x,y) + F_{2,1}(x,y) = \frac{\left(1 + \dfrac{y}{1-x^2}\right)\left(1 + \dfrac{xy}{1-x^2}\right)}{1 - \dfrac{xy^2}{(1-x^2)^2}}.$$

From the first of (3.2), we get

$$\begin{aligned} F_{2,0}(x,y) &= \left(1 + \frac{y}{1-x^2}\right)\sum_{r=0}^{\infty}\frac{x^r y^{2r}}{(1-x^2)^{2r}} \\ &= \sum_{r=0}^{\infty} x^r y^{2r} \sum_{s=0}^{\infty}\binom{2r+s-1}{s}x^{2s} \\ &\quad + \sum_{r=0}^{\infty} x^r y^{2r+1}\sum_{s=0}^{\infty}\binom{2r+s}{s}x^{2s} \end{aligned}$$

$$\text{(3.4)}\qquad = \sum_{\substack{n=0 \\ r+2s=n}}^{\infty}\binom{2r+s-1}{s}x^n y^{2r} + \sum_{\substack{n=0 \\ r+2s=n}}^{\infty}\binom{2r+s}{s}x^n y^{2r+1}.$$

Since

$$F_{2,0}(x,y) = \sum_{n,k=0}^{\infty} f_{2,0}(n,k)x^n y^k,$$

it follows from (3.4) that

$$f_{2,0}(n,k) = \binom{k+s-1}{s}, \tag{3.5}$$

where

$$s = \begin{cases} \frac{1}{2}\left(n - \frac{1}{2}(k)\right) & (k \text{ even}) \\ \frac{1}{2}\left(n - \frac{1}{2}(k-1)\right) & (k \text{ odd}), \end{cases}$$

that is,

$$s = \frac{1}{2}\left(n - \left[\frac{1}{2}(k)\right]\right). \tag{3.6}$$

Similarly,

$$F_{2,1}(x,y) = \sum_{r=0}^{\infty} x^{r+1}y^{2r+1}\sum_{s=0}^{\infty}\binom{2r+s}{s}x^{2s}$$

$$+ \sum_{r=0}^{\infty} x^{r+1}y^{2r+2}\sum_{s=0}^{\infty}\binom{2r+s+1}{s}x^{2s}$$

$$= \sum_{\substack{n=1 \\ r+2s+1=n}}^{\infty}\binom{2r+s}{s}x^n y^{2r+1} + \sum_{\substack{n=1 \\ r+2s+1=n}}^{\infty}\binom{2r+s+1}{s}x^n y^{2r+2}. \tag{3.7}$$

Since

$$F_{2,1}(x,y) = \sum_{n,k=1}^{\infty} f_{2,1}(n,k)x^n y^k,$$

it follows from (3.7) that

$$f_{2,1}(n,k) = \binom{k+s-1}{s}, \tag{3.8}$$

where

$$s = \begin{cases} \frac{1}{2}\left(n - \frac{1}{2}(k+1)\right) & (k \text{ odd}) \\ \frac{1}{2}\left(n - \frac{1}{2}(k)\right) & (k \text{ even}), \end{cases}$$

that is

$$s = \frac{1}{2}\left(n - \left[\frac{1}{2}(k+1)\right]\right). \tag{3.9}$$

Hence, we can combine (3.5), (3.6), (3.8), (3.9) in the formula

$$f_{2,i}(n,k) = \binom{k+s-1}{s} \qquad (i = 0, 1), \tag{3.10}$$

where

$$s = \frac{1}{2}\left(n - \left[\frac{1}{2}(k+i)\right]\right). \tag{3.11}$$

For $y = 1$, (3.4) reduces to

$$F_{2,0}(x,1) = \sum_{n=0}^{\infty} x^n \sum_{2s \le n}\left\{\binom{2r+s-1}{s} + \binom{2r+s}{s}\right\},$$

so that

$$f_{2,0}(n) = \sum_{2s \le n}\left\{\binom{2n-3s-1}{s} + \binom{2n-3s}{s}\right\}. \tag{3.12}$$

Similarly, (3.7) yields

$$F_{2,1}(x,1) = \sum_{n=1} x^n \sum_{r+2s+1=n} \left\{ \binom{2r+s}{s} + \binom{2r+s+1}{s} \right\},$$

which implies

$$f_{2,1}(n) = \sum_{2s \le n-1} \left\{ \binom{2n-3s-2}{s} + \binom{2n-3s-1}{s} \right\}. \tag{3.13}$$

We can combine (3.12) and (3.13) in the single formula

$$f_{2,i}(n) = \sum_{2s \le n-i} \left\{ \binom{2n-3s-i-1}{s} + \binom{2n-3s-i}{s} \right\} \quad (i=0,\ 1). \tag{3.14}$$

It follows from (3.14) that

$$f_2(n) = \sum_{2s \le n} \left\{ \binom{2n-3s}{s} + 2\binom{2n-3s-1}{s} + \binom{2n-3s-2}{s} \right\}. \tag{3.15}$$

## SECTION 4

For arbitrary $m \ge 1$, we have, by (2.8),

$$f_{m,j}(n,k,a) = \sum_{\substack{i=0 \\ i \ne j}}^{m-1} f_{m,i}(n-a,k-1) \qquad [k > 0,\ a \equiv j \pmod{m}]$$

together with

$$\begin{cases} f_{m,0}(n,1,a) = \delta_{n,a} & [a \equiv 0 \pmod{m}] \\ f_{m,0}(n,0,a) = \delta_{n0}\delta_{a0}. \end{cases}$$

It follows that

$$\begin{cases} F_{m,0}(x,y,a) = \delta_{a,0} + x^a y + x^a y \sum_{i=1}^{m-1} F_{m,i}(x,y) & [a \equiv 0 \pmod{m}] \\ F_{m,j}(x,y,a) = x^a y \sum_{\substack{i=0 \\ i \ne j}}^{m-1} F_{m,i}(x,y) & [1 \le j < m;\ a \equiv j \pmod{m}]. \end{cases}$$

Summing over $a$ we get

$$\begin{cases} F_{m,0}(x,y) = 1 + \dfrac{y}{1-x^m} + \dfrac{y}{1-x^m} \sum_{i=1}^{m-1} F_{m,i}(x,y) \\ F_{m,j}(x,y) = \dfrac{x^j y}{1-x^m} \sum_{\substack{i=0 \\ i \ne j}}^{m-1} F_{m,i}(x,y) \quad (1 \le j < m). \end{cases} \tag{4.1}$$

Since

$$\sum_{\substack{i=0 \\ i \ne j}}^{m-1} F_{m,i}(x,y) = F_m(x,y) - F_{m,j}(x,y),$$

(4.1) becomes

$$\left\{\begin{aligned}\left(1 + \frac{y}{1 - x^m}\right)F_{m,0}(x,y) &= 1 + \frac{y}{1 - x^m} + \frac{y}{1 - x^m}F_m(x,y)\\ \left(1 + \frac{x^j y}{1 - x^m}\right)F_{m,j}(x,y) &= \frac{x^j y}{1 - x^m}F_m(x,y) \qquad (1 \le j < m).\end{aligned}\right. \tag{4.2}$$

This in turn gives

$$\left\{\begin{aligned}F_{m,0}(x,y) &= 1 + \frac{\dfrac{y}{1 - x^m}}{1 + \dfrac{y}{1 - x^m}}F_m(x,y)\\ F_{m,j}(x,y) &= \frac{\dfrac{x^j y}{1 - x^m}}{1 + \dfrac{x^j y}{1 - x^m}}F_m(x,y) \qquad (1 \le j < m).\end{aligned}\right.$$

Hence, by adding together these equations, we get

$$\left\{1 - \sum_{j=0}^{m-1}\frac{\dfrac{x^j y}{1 - x^m}}{1 + \dfrac{x^j y}{1 - x^m}}\right\}F_m(x,y) = 1. \tag{4.3}$$

For brevity, put $z = y/(1 - x^m)$, so that (4.3) reduces to

$$\left\{1 - \sum_{j=0}^{m-1}\frac{x^j z}{1 + x^j z}\right\}F_m(x,y) = 1. \tag{4.4}$$

Put

$$P_m(z) = P_m(z,x) = \prod_{j=0}^{m-1}(1 + x^j z). \tag{4.5}$$

It is well-known that

$$P_m(z) = \sum_{j=0}^{m}\begin{bmatrix}m\\ j\end{bmatrix}x^{\frac{1}{2}j(j-1)}z^j, \tag{4.6}$$

where

$$\begin{bmatrix}m\\ j\end{bmatrix} = \frac{(1 - x^m)(1 - x^{m-1}) \ldots (1 - x^{m-j+1})}{(1 - x)(1 - x^2) \ldots (1 - x^j)}.$$

Moreover, it follows from (4.5) that

$$\frac{zP_m'(z)}{P_m(z)} = \sum_{j=0}^{m-1}\frac{x^j z}{1 + x^j z}.$$

Thus (4.4) becomes

$$\left\{1 - \frac{zP_m'(z)}{P_m(z)}\right\} F_m(x,y) = 1,$$

and therefore

$$(4.7)\qquad F_m(x,y) = \frac{P_m(z)}{Q_m(z)} \qquad \left(z = \frac{y}{1 - x^m}\right),$$

where

$$(4.8)\qquad Q_m(z) = P_m(z) - zP_m'(z) = \sum_{j=0}^{m} (1 - j)\begin{bmatrix} m \\ j \end{bmatrix} x^{\frac{1}{2}j(j-1)} z^j.$$

For example, for $m = 2$, (4.7) gives

$$(4.9)\qquad F_2(x,y) = \frac{(1 + z)(1 + xz)}{1 - xz^2} \qquad \left(z = \frac{y}{1 - x^2}\right),$$

while, for $m = 3$, we get

$$(4.10)\qquad F_3(x,y) = \frac{(1 + z)(1 + xz)(1 + yz)}{1 - (x + x^2 + x^3)z^2 - 2x^3z^3} \qquad \left(z = \frac{y}{1 - x^3}\right).$$

## SECTION 5

A few words may be added about the limiting case $m = \infty$. We take $|x| < 1$ so that $x^m \to 0$ and

$$z = \frac{y}{1 - x^m} \to y.$$

Thus (4.3) becomes

$$(5.1)\qquad \left\{1 - \sum_{j=0}^{\infty} \frac{x^j y}{1 + x^j y}\right\}\left\{1 + \sum_{n,k=1}^{\infty} f_\infty(n,k) x^n y^k\right\} = 1.$$

On the other hand, the condition

$$a_i \not\equiv a_{i+1} \pmod{m} \qquad (i = 1, 2, \ldots, k - 1)$$

becomes

$$(5.2)\qquad a_i \neq a_{i+1} \qquad (i = 1, 2, \ldots, k - 1).$$

In the notation of [1], the number of solutions in nonnegative integers of $n = a_1 + \cdots + a_k$ and (5.2) is denoted by $\overline{c}(n,k)$ and it is proved that

$$(5.3)\qquad 1 + \sum_{n,k=1}^{\infty} \overline{c}(n,k) x^n y^k = \left\{1 + \sum_{j=1}^{\infty} (-1)^j \frac{y^j}{1 - x^j}\right\}^{-1}.$$

Clearly,

$$(5.4)\qquad f_\infty(n,k) = \overline{c}(n,k).$$

To verify that (5.1) and (5.3) are equivalent, we take

$$1 - \sum_{j=0}^{\infty} \frac{x^j y}{1 + x^j y} = 1 - \sum_{j=0}^{\infty} x^j y \sum_{s=0}^{\infty} (-1)^s x^{sj} y^s = 1 + \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} (-1)^k x^{jk} y^k$$
$$= 1 + \sum_{k=1}^{\infty} (-1)^k \frac{y^k}{1 - x^k}.$$

## REFERENCE

1. L. Carlitz. "Restricted Compositions." *The Fibonacci Quarterly* 14, No. 3 (1976):254-264.


*****

# CHEBYSHEV AND FERMAT POLYNOMIALS FOR DIAGONAL FUNCTIONS


A. F. HORADAM

*University of New England, Armidale, N.S.W., Australia*


## INTRODUCTION

Jaiswal [3] and the author [1] examined rising diagonal functions of Chebyshev polynomials of the second and first kinds, respectively. Also, in [2], the author investigated rising and descending functions of a wide class of sequences satisfying certain criteria. Excluded from consideration in [2] were the Chebyshev and Fermat polynomials that did not satisfy the restricting criteria.

The object of this paper is to complete the above articles by studying *descending* diagonal functions for the Chebyshev polynomials in Part I, and *both* rising and descending diagonal functions for the Fermat polynomials in Part II.

Chebyshev polynomials $T_n(x)$ of the second kind are defined by

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x) \qquad T_0(x) = 2,\ T_1(x) = 2x \qquad (n \geq 0), \tag{1}$$

while Chebyshev polynomials $U_n(x)$ of the first kind are defined by

$$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x) \qquad U_0(x) = 1,\ U_1(x) = 2x \qquad (n \geq 0). \tag{2}$$

Often we write $x = \cos\theta$ to obtain trigonometrical sequences.

## PART I

### DESCENDING DIAGONAL FUNCTIONS FOR $T_n(x)$

From (1), we obtain

$$\left\{\begin{array}{l}
T_0(x) = 2 \\
T_1(x) = 2x \\
T_2(x) = 4x^2 - 2 \\
T_3(x) = 8x^3 - 6x \\
T_4(x) = 16x^4 - 16x^2 + 2 \\
T_5(x) = 32x^5 - 40x^3 + 10x \\
T_6(x) = 64x^6 - 96x^4 + 36x^3 - 2 \\
T_7(x) = 128x^7 - 224x^5 + 112x^3 - 14x \\
T_8(x) = 256x^8 - 512x^6 + 320x^4 - 64x^2 + 2 \\
T_9(x) = 512x^9 - 1152x^7 + 864x^5 - 240x^3 + 18x \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}\right. \tag{3}$$