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ON SOME EXTENSIONS OF THE WANG-CARLITZ IDENTITY

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ABSTRACT

Two theorems are presented which generalize a recent Wang [6]-Carlitz [1] result. In addition, we also obtain its Abel analogue. The method of proof is dependent upon some of our recent work [2].

I

Wang [6] proved the expansion

$$\sum_{k=1}^{r+1}\binom{r+1}{k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_j>0}}\prod_{m=1}^{k}(i_m+1)=\binom{n+2r+1}{2r+1}. \tag{1.1}$$

Recently, Carlitz [1] extended (1.1) to

$$\sum_{k=0}^{r+1}\binom{r+1}{k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_j>0}}\prod_{m=1}^{k}\binom{i_m+a}{i_m}=\binom{n+ar+r+a}{n}. \tag{1.2}$$

Theorems 1 and 2 in this paper treat a number of different generalizations of (1.2). In particular, a special case of Theorem 1 gives the new expression:

$$\begin{aligned}&\sum_{k=0}^{r+1}\binom{r+1}{k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_j>0}}\prod_{m=1}^{k}\frac{(a+1)}{(a+1+ti_m)}\binom{a+ti_m+i_m}{i_m}\\ &=\frac{(a+1)(r+1)}{(a+1)(r+1)+tn}\binom{ar+r+a+tn+n}{n}.\end{aligned} \tag{1.3}$$

Letting $t = 0$ in (1.3) yields (1.2).

We also present the Abel analogue of (1.3):

$$(1.4)\quad \sum_{k=0}^{r+1}\binom{r+1}{k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_j>0}}\prod_{m=1}^{k}\frac{(\alpha)(\alpha+\ell i_m)^{i_m-1}}{i_m!}=\frac{(\alpha)(r+1)(\alpha r+r+\ell n)^{n-1}}{n!}.$$

(1.3) and (1.4) are special cases of a number of classes of functions including some well-known orthogonal polynomials. These are considered in the following theorem.

Theorem 1: For $\alpha,\beta,\ell,\ell',t$ complex numbers, r and n nonnegative integers, and s a positive integer

$$(1.5)\qquad \sum_{k=0}^{r}\binom{r}{k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_j>0}}\prod_{m=1}^{k}\theta_{i_m}^{\alpha,\beta}(x)=\theta_n^{\alpha r,\beta r}(x)$$

where $(a)_k=\Gamma(a+k)/\Gamma(a)$, quotient of gamma functions,

$[n/s]$ is the greatest integer notation, and

$\theta_n^{\alpha,\beta}(x)$ can assume any of the ten functions:

(1.6) $A_n^{\alpha,\beta}(x)$

$$=\frac{(\alpha)_{tn+n}}{n!(\alpha+1)_{tn}}\sum_{p=0}^{[n/s]}\frac{(-n)_{sp}(\alpha+tn+n)_{\ell p}(\beta)_{\ell' p}\{(\alpha+stp+sp+\ell p)/\alpha\}x^p}{p!(\alpha+1+tn)_{sp+\ell p}(\beta+1)_{\ell' p-p}}$$

(1.7) $B_n^{\alpha,\beta}(x)$

$$=\frac{(\beta)(\alpha)_{tn+n}}{n!(\alpha+1)_{tn}}\sum_{p=0}^{[n/s]}\frac{(-n)_{sp}(\alpha+tn+n)_{\ell p}(\beta+\ell' p)^{p-1}\{(\alpha+stp+sp+\ell p)/\alpha\}x^p}{p!(\alpha+1+tn)_{sp+\ell p}}$$

(1.8) $C_n^{\alpha,\beta}(x)$

$$=\frac{(\alpha+\ell n)^{n-1}}{n!}\sum_{p=0}^{[n/s]}\frac{(-1)^{sp}(-n)_{sp}(\beta)_{\ell' p}\{\alpha+\ell sp\}x^p}{p!(\alpha+\ell n)^{sp}(\beta+1)_{\ell' p-p}}$$

(1.9) $D_n^{\alpha,\beta}(x)$

$$=\frac{(\beta)(\alpha+\ell n)^{n-1}}{n!}\sum_{p=0}^{[n/s]}\frac{(-1)^{sp}(-n)_{sp}(\beta+\ell' p)^{p-1}\{\alpha+\ell sp\}x^p}{p!(\alpha+\ell n)^{sp}}$$

(1.10) $E_n^{\alpha,\beta}(x)$

$$=\frac{1}{(\alpha+\ell n)n!}\sum_{p=0}^{[n/s]}\frac{(-1)^{sp}(-n)_{sp}(\beta)_{n\ell'-sp\ell'}(\alpha+\ell n)^p\{\alpha+\ell n-\ell sp\}x^{n-sp}}{p!(\beta+1)_{n\ell'-n-sp\ell'+sp}}$$

(1.11) $F_n^{\alpha,\beta}(x)$

$$=\frac{(\beta)}{(\alpha+\ell n)n!}\sum_{p=0}^{[n/s]}\frac{(-1)^{sp}(-n)_{sp}(\alpha+\ell n)^p(\beta+\ell' n-\ell' sp)^{n-sp-1}\{\alpha+\ell n-\ell sp\}x^{n-sp}}{p!}$$

$$G_n^{\alpha,\beta}(x) = \frac{1}{n!}\sum_{p=0}^{\infty}\frac{(\alpha + tp)_{\ell n}(\beta)_{\ell' p}x^p}{p!(1+\alpha+tp)_{\ell n - n}(\beta+1)_{\ell' p - p}} \tag{1.12}$$

$$H_n^{\alpha,\beta}(x) = \frac{(\beta)}{n!}\sum_{p=0}^{\infty}\frac{(\alpha + tp)_{\ell n}(\beta + \ell' p)^{p-1}x^p}{p!(\alpha+tp+1)_{\ell n - n}} \tag{1.13}$$

$$I_n^{\alpha,\beta}(x) = \frac{1}{n!}\sum_{p=0}^{\infty}\frac{(\alpha + tp + \ell n)^{n-1}(\beta)_{\ell' p}\{\alpha + tp\}x^p}{p!(\beta+1)_{\ell' p - p}} \tag{1.14}$$

$$J_n^{\alpha,\beta}(x) = \frac{(\beta)}{n!}\sum_{p=0}^{\infty}\frac{(\alpha + tp + \ell n)^{n-1}(\beta + \ell' p)^{p-1}\{\alpha + tp\}x^p}{p!}. \tag{1.15}$$

Proof of (1.6): From Theorem 4b [2, p. 708],

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{v^n(\alpha)_{tn+n}}{n!(\alpha+1)_{tn}}\sum_{p=0}^{[n/s]}\frac{(-n)_{sp}(\alpha+tn+n)_{\ell p}(\beta)_{\ell' p}\{(\alpha+tsp+sp+\ell p)/\alpha\}x^p}{p!(tn+\alpha+1)_{sp+\ell p}(\beta+1)_{\ell' p - p}}\\ = (1-z)^{\alpha}\sum_{p=0}^{\infty}\frac{(\beta)_{\ell' p}x^p z^{sp}(1-z)^{\ell p}}{p!(\beta+1)_{\ell' p - p}}\end{aligned} \tag{1.16}$$

where $v(1-z)^{t+1} = -z$, $v(0) = 0$.

Hence,

$$\sum_{k=0}^{r}\binom{r}{k}\left\{\sum_{n=1}^{\infty}v^n A_n^{\alpha,\beta}(x)\right\}^k = (1-z)^{\alpha r}\sum_{p=0}^{\infty}\frac{(\beta r)_{\ell' p}x^p z^{sp}(1-z)^{\ell p}}{p!(\beta r+1)_{\ell' p - p}}. \tag{1.17}$$

(1.17) may be expressed as

$$\sum_{n=0}^{\infty}v^n\sum_{k=0}^{r}\binom{r}{k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_j>0}}\prod_{m=1}^{k}A_{i_m}^{\alpha,\beta}(x) = \sum_{n=0}^{\infty}v^n A_n^{\alpha r,\beta r}(x). \tag{1.18}$$

Comparing coefficients on both sides gives the required (1.6).

Proof of (1.7): From Theorem 4b [2, p. 708],

$$\left\{\sum_{n=1}^{\infty}v^n B_n^{\alpha,\beta}(x)\right\}^k = \left\{(1-z)^{\alpha}\sum_{p=0}^{\infty}\frac{(\beta+\ell' p)^{p-1}x^p z^{sp}(1-z)^{\ell p}}{p!} - 1\right\}^k. \tag{1.19}$$

(1.19) is obtained by modifying the arbitrary sequence $\{e_p\}$ to be

$(\beta + \ell' p)^{p-1}$.

Hence,

$$\sum_{k=0}^{r}\binom{r}{k}\left\{\sum_{n=1}^{\infty}v^n B_n^{\alpha,\beta}(x)\right\}^k \tag{1.20}$$

$$= \beta r(1-z)^{\alpha r}\sum_{p=0}^{\infty}\frac{(\beta r+\ell' p)^{p-1}x^p z^{sp}(1-z)^{\ell p}}{p!} \tag{1.21}$$

$$(1.22) \qquad = \sum_{n=0}^{\infty} v^n B_n^{\alpha r,\, \beta r}(x)$$

$$(1.23) \qquad = \sum_{n=0}^{\infty} v^n \sum_{k=0}^{r} \binom{r}{k} \sum_{\substack{i_1 + \cdots + i_k = n \\ i_j > 0}} \prod_{m=1}^{k} B_{i_m}^{\alpha,\beta}(x).$$

Comparing coefficients gives (1.7).

The proofs of (1.6) and (1.7) are the procedures adopted in the above cases with suitable modifications. We are initially required to establish generating functions.

Proof of (1.8): Theorem 2b [2, p. 704] will give

$$(1.24) \qquad \sum_{n=0}^{\infty} u^n C_n^{\alpha,\beta}(x) = \exp(\alpha z) \sum_{p=0}^{\infty} \frac{(\beta)_{\ell' p}\, x^p z^{sp}}{p!\,(\beta + 1)_{\ell' p - p}}.$$

Proof of (1.9): Theorem 2b [2, p. 704] yields

$$(1.25) \qquad \sum_{n=0}^{\infty} u^n D_n^{\alpha,\beta}(x) = \exp(\alpha z) \sum_{p=0}^{\infty} \frac{(\beta + \ell' p)^{p-1} x^p z^{sp}}{p!}.$$

Proof of (1.10): Using Theorem 2d [2, p. 704], one may obtain

$$(1.26) \qquad \sum_{n=0}^{\infty} w^n E_n^{\alpha,\beta}(x) = \exp(\alpha z) \sum_{p=0}^{\infty} \frac{(\beta)_{\ell' p}\, x^p z^{p/s}}{p!\,(\beta + 1)_{\ell' p - p}},$$

where $w = z^{1/s} \exp(-\ell z)$.

Proof of (1.11): From Theorem 2d [2, p. 704],

$$(1.27) \qquad \sum_{n=0}^{\infty} w^n F_n^{\alpha,\beta}(x) = \exp(\alpha z) \sum_{p=0}^{\infty} \frac{(\beta + \ell' p)^{p-1} x^p z^{p/s}}{p!}.$$

Proof of (1.12): It may be shown that

$$(1.28) \qquad \sum_{n=0}^{\infty} \xi^n G_n^{\alpha,\beta}(x) = (1 - z)^{\alpha} \sum_{p=0}^{\infty} \frac{(\beta)_{\ell' p}\, (1 - z)^{tp} x^p}{p!\,(\beta + 1)_{\ell' p - p}},$$

where $\xi(1 - z)^{\ell} = -z$.

Proof of (1.13): One may derive

$$(1.29) \qquad \sum_{n=0}^{\infty} \xi^n H_n^{\alpha,\beta}(x) = (1 - z)^{\alpha} \sum_{p=0}^{\infty} \frac{(\beta + \ell' p)^{p-1} x^p}{p!}.$$

Proof of (1.14): The generating function of $I_n^{\alpha,\beta}(x)$ is

$$\sum_{n=0}^{\infty} u^n I_n^{\alpha,\beta}(x) = \exp(\alpha z)\sum_{p=0}^{\infty}\frac{(\beta)_{\ell' p}\, x^p \exp(ptz)}{p!\,(\beta+1)_{\ell' p - p}}. \tag{1.30}$$

Proof of (1.15): It may be proved that

$$\sum_{n=0}^{\infty} u^n J_n^{\alpha,\beta}(x) = \exp(\alpha z)\sum_{p=0}^{\infty}\frac{(\beta+\ell' p)^{p-1} x^p \exp(ptz)}{p!}. \tag{1.31}$$

II

A second generalization of the Carlitz result given by our equation (1.2) is

$$\sum_{k=0}^{r+1}\binom{r+1}{k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_j>0}}\prod_{m=1}^{k}\binom{a+ti_m+i_m}{i_m} \tag{2.1}$$
$$= \frac{(ar+r+a+1)_{tn+n}}{n!\,(ar+r+a+2)_{tn}}\sum_{p=0}^{n}\frac{(-n)_p\,(r+1)_p\,(-t)^p\{(ar+r+a+1+tp+p)/(ar+r+a+1)\}}{p!\,(ar+r+a+2+tn)_p}.$$

For $t = 0$, the polynomial reduces to unity and (1.2) presents itself.

We also have the Abel analogue of (2.1), which assumes the form

$$\sum_{k=0}^{r+1}\binom{r+1}{k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_j>0}}\prod_{m=1}^{k}\frac{(\alpha+\ell i_m)^{i_m}}{i_m!} \tag{2.2}$$
$$= \frac{(\alpha r+\ell n)^{n-1}}{n!}\sum_{p=0}^{n}\frac{(-\ell)^p(-n)_p(r)_p\{\alpha r+p\}}{p!\,(\alpha r+\ell n)^p}.$$

Now both (2.1) and (2.2) are particular cases of Theorem 2.

Theorem 2: For α, β, ℓ, t complex numbers, r and n nonnegative integers,

$$\text{a.}\quad \sum_{k=0}^{r}\binom{r}{k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_j>0}}\prod_{m=1}^{k}R_{i_m}^{\alpha,\beta}(t) = R^{\alpha r,\,\beta r+r-1}(t) \tag{2.3}$$

and

$$\text{b.}\quad \sum_{k=0}^{r}\binom{r}{k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_j>0}}\prod_{m=1}^{k}S_{i_m}^{\alpha,\beta}(\ell) = S_n^{\alpha r,\,\beta r+r-1}(\ell), \tag{2.4}$$

where

$$R_n^{\alpha,\beta}(t) = \frac{(\alpha)_{tn+n}}{n!\,(\alpha)_{tn}}\sum_{p=0}^{n}\frac{(-n)_p(-t)^p(\beta)_p}{p!\,(\alpha+tn)_p}, \tag{2.5}$$

$$S_n^{\alpha,\beta}(\ell) = \frac{(\alpha+\ell n)^n}{n!}\sum_{p=0}^{n}\frac{(-\ell)^p(-n)_p(\beta)_p}{p!\,(\alpha+\ell n)^p}. \tag{2.6}$$

Proof of Theorem 2(a): With the aid of Theorem 4a [2, p. 708],

$$\sum_{n=0}^{\infty}R_n^{\alpha,\beta}(t)y^n = \frac{(1-z)^{\alpha}}{(1+tz)^{\beta+1}}, \tag{2.7}$$

where $y(1 - z)^{t+1} = -z$. Note the misprint in equation (4.3) [2, p. 709], in the definition of $C_n(x)$. The factor $(\alpha + 1 + mn - n)_{sk+\ell k}$ should read

$$(\alpha + mn - n)_{sk+\ell k}.$$

Hence

$$\sum_{k=0}^{r}\binom{r}{k}\left\{\sum_{n=1}^{\infty} y^n R_n^{\alpha,\beta}(t)\right\}^k = \frac{(1-z)^{\alpha r}}{(1+tz)^{\beta r + r}} \tag{2.8}$$

$$= \sum_{n=0}^{\infty} y^n R_n^{\alpha r, \beta r + r - 1}(t). \tag{2.9}$$

Comparing coefficients and simplifying gives (2.3).

Proof of Theorem 2(b): Using Theorem 2a [2, p. 704],

$$\sum_{n=0}^{\infty} w^n S_n^{\alpha,\beta}(\ell) = \frac{\exp(\alpha z)}{(1-\ell z)^{\beta+1}} \tag{2.10}$$

where $w = z\exp(-z\ell)$. Thus,

$$\sum_{k=0}^{r}\binom{r}{k}\left\{\sum_{n=1}^{\infty} w^n S_n^{\alpha,\beta}(\ell)\right\}^k = \frac{\exp(\alpha z r)}{(1-\ell z)^{\beta r + r}}. \tag{2.11}$$

Proceeding as in part a gives the required (2.4).

III. SPECIAL CASES

It is of interest to note that a number of well-known polynomials form special cases of Theorem 1.

1. Putting $x = (1 - y)/2$, $s = 1$, $\ell = 0$, $\ell' = 1$, one may express $A_n^{\alpha,\beta}(x)$ in (1.6) as a Jacobi polynomial of the form

$$\frac{\alpha}{\alpha + tn + n}\left\{P_n^{\alpha+tn,\beta-1-\alpha-tn-n}(y) + \frac{(t+1)y}{\alpha}\frac{d}{dy}P_n^{\alpha+tn,\beta-1-\alpha-tn-n}(y)\right\} \tag{3.1}$$

where the Jacobi polynomial is defined in [4, p. 170].

2. Letting $\ell = 0$, $\ell' = 0$, $s = 1$, one may express $B_n^{\alpha,\beta}(x)$ from (1.7) as

$$\frac{\alpha}{\alpha + tn + n}\left\{L_n^{\alpha+tn}(\beta x) + \frac{(t+1)x}{\alpha}\frac{d}{dx}L_n^{\alpha+tn}(\beta x)\right\}, \tag{3.2}$$

where the Laguerre polynomial is defined in [4, p. 188]. Hence, one may view $B_n^{\alpha,\beta}(x)$ as a generalized Laguerre polynomial.

3. $E_n^{\alpha,\beta}(x)$ may be viewed as a generalized Laguerre polynomial with the degree of the polynomial incorporated in the argument. In the special case for $x = 1/y$, $s = 1$, $\ell' = 1$,

$$E_n^{\alpha,\beta}(x) = \frac{(-1)^n}{\alpha + \ell n}\left[\alpha y^{-n} L_n^{-\beta-n}[y(\alpha+\ell n)] - \ell y\frac{d}{dy}\left[y^{-n}L_n^{-\beta-n}[y(\alpha+\ell n)]\right]\right]. \tag{3.3}$$

4. $F_n^{\alpha,\beta}(x)$ may be looked upon as a generalization of the generalized Hermite polynomial defined by Gould et al. [5, p. 58, eqn. (6.2)], and others. See also [2] for properties of this polynomial. The generalized Hermite is defined as

$$g_n^s(x,\lambda) = H_{n,s}(x,\lambda) = \sum_{k=0}^{[n/s]} \frac{n!\lambda^k x^{n-sk}}{k!(n-sk)!}. \tag{3.4}$$

Letting $\ell' = 0$ in (1.11), one obtains

$$F_n^{\alpha,\beta}(x) = \frac{1}{n!(\alpha+\ell n)}\left[\alpha H_{n,s}(\beta x, \alpha+\ell n) + \ell x \frac{d}{dx} H_{n,s}(\beta x, \alpha+\ell n)\right]. \tag{3.5}$$

Further, putting $\ell = 0$, one obtains a single term on the right-hand side. See also [3] for bilinear generating functions and other expansions for the generalized Hermite polynomial.

5. For the special case $\ell' = 1$, $G_n^{\alpha,\beta}(x)$ may be expressed as a general polynomial of the type

$$\frac{(1-x)^{-\beta-n}}{n!}\sum_{k=0}^{n}(-\beta-n)_k x^k \sum_{p=0}^{k} \frac{(\alpha+tp)_{\ell n}(\beta)_p(-1)^p}{(k-p)!p!(1+\alpha+tp)_{\ell n-n}(1+\beta+n-k)_p}. \tag{3.6}$$

6. $H_n^{\alpha,\beta}(x)$ may be considered as a generalization of a polynomial considered by Gould et al. [5], defined in equation (3.2), p. 53. For the special case $\ell' = 0$,

$$H_n^{\alpha,\beta}(x) = \frac{\exp(\beta x)}{n!}\sum_{k=0}^{n}(-1)^k(\beta x)^k \sum_{p=0}^{k}\frac{(-1)^p(\alpha+tp)_{\ell n}}{(k-p)!p!(\alpha+tp+1)_{\ell n-n}}. \tag{3.7}$$

Further, for $\ell = 0$, $H_n^{\alpha,\beta}(x)\exp(-\beta x)$ gives essentially the polynomial of Gould et al.

The polynomials considered in this paper appear to possess interesting common algebraic properties. One of them is that they all arise from representations of the same group. We shall have occasion to discuss group-theoretical properties of these polynomials elsewhere.

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