There is a close connection between the Golden Ratio and the Fibonacci Sequence—1, 1, 2, 3, 5, 8, 13, 21, ... Each number is obtained by adding the two numbers just previous to it. This numerical sequence is named after the thirteenth-century Italian mathematician Leonardo Pisano, who discovered it while solving a problem on the breeding of rabbits. Ratios of successive pairs of some initial numbers give the following values:

1/1 = 1.000; 1/2 = 0.500; 2/3 = 0.666...; 3/5 = 0.600; 5/8 = 0.625;
8/13 = 0.615...; 13/21 = 0.619...; 21/34 = 0.617...; 34/55 = 0.618...;
55/89 = 0.618... .

Thereafter, the ratio reaches a constant that is almost equivalent to the Golden Ratio. Such a ratio has been detected in most plants with alternate (spiral) phyllotaxis, because any two consecutive leaves subtend a Fibonacci angle approximating 317.5 degrees. Thus, many investigators of phyllotaxis identify the involvement of Fibonacci series on foliar arrangement, the most recent being Mitchison [6].

(please turn to page 384)

A RECURRENCE RELATION FOR GENERALIZED MULTINOMIAL COEFFICIENTS

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1. INTRODUCTION

Gould [2] has defined Fontené-Ward multinomial coefficients by

$$\left\{ {n \atop s_1,\ s_2,\ \ldots,\ s_r} \right\} = u_n!/u_{s_1}!u_{s_2}!\ \ldots\ u_{s_r}!$$

where $\{u_n\}$ is an arbitrary sequence of real or complex numbers such that

$$u_n \neq 0 \text{ for } n \geq 1,$$

$$u_0 = 0,$$

$$u_1 = 1,$$

and $\quad u_n! = u_n u_{n-1} \ldots u_1,$

with $\quad u_0! = 1.$

These are a generalization of ordinary multinomial coefficients for which there is a recurrence relation

$$\left({n \atop s_1,\ \ldots,\ s_r} \right) = \sum_{j=1}^{r} \left({n-1 \atop s_1 - \delta_{1j},\ \ldots,\ s_r - \delta_{rj}} \right)$$

as in Hoggatt and Alexanderson [4].

Hoggatt [3] has also studied Fontené-Ward coefficients when $r = 2$ and $\{u_n\} = \{F_n\}$, the sequence of Fibonacci numbers. We propose to consider the case where the u_n are elements which satisfy a linear homogeneous recurrence relation of order r.

2. THE COEFFICIENTS

We consider r basic sequences, $\left\{u_{s,n}^{(r)}\right\}$, which satisfy a recurrence relation of order r:

$$U_{s,n}^{(r)} = \sum_{j=1}^{r} (-1)^{j+1} P_{rj} U_{s,n-j}^{(r)}, \; n > r; \; U_{s,n}^{(r)} = \delta_{sn}, \; 1 \le n \le r,$$

where δ_{sn} is the Kronecker delta and the P_{rj} are arbitrary integers. We designate $\left\{U_{r,n}^{(r)}\right\}$ as the fundamental sequence by analogy with Lucas' second-order fundamental sequence $\left\{U_{2,n}^{(2)}\right\}$. Since this sequence is used frequently, we let

$$\left\{U_{r,n+r}^{(r)}\right\} = \left\{u_n^{(r)}\right\}$$

for convenience of notation.

Note that the terms "fundamental" and "basic" follow from the nature of these sequences as expounded in Jarden [5] and Bell [1], respectively.

Let $M_n^{(r)}$ denote the square matrix of order r:

$$M_n^{(r)} = \left[U_{j,n+i}^{(r)}\right], \; 1 \le i, \; j \le r$$

wherein i refers to the rows and j to the columns.

<u>*Lemma*</u>: $u_n^{(r)} = \sum_{j=0}^{r-1} U_{r-j,r+m}^{(r)} u_{n-m-j}^{(r)}$.

<u>*Proof*</u>: It is easily proved by induction that

$$M_n^{(r)} = M_m^{(r)} M_{n-m}^{(r)},$$

and so from equating the elements in the last row and last column we get

$$u_n^{(r)} = U_{r,r+n}^{(r)} = \sum_{j=0}^{r-1} U_{r-j,r+m}^{(r)} U_{r,r+n-m-j}^{(r)}$$

$$= \sum_{j=0}^{r-1} U_{r-j,r+m}^{(r)} u_{n-m-j}^{(r)}.$$

We now define Fibonacci multinomial coefficients by

$$\left\{\begin{matrix} n \\ s_1, \ldots, s_r \end{matrix}\right\}_u = u_n^{(r)}!/u_{s_1}^{(r)} \ldots u_{s_r}^{(r)},$$

such that $n = \sum_{i=1}^{r} s_i$.

Thus, when $r = 2$, we have the Fibonacci binomial coefficients

$$\left\{\begin{matrix} n \\ s_1, s_2 \end{matrix}\right\}_u = \frac{u_n^{(r)}!}{u_{s_1}^{(2)}!u_{s_2}^{(2)}!} = \frac{u_n^{(2)}!}{u_{s_1}^{(2)}!u_{n-s_1}^{(2)}!} = \left\{\begin{matrix} n \\ s_1 \end{matrix}\right\}.$$

We next seek the recurrence relation for these Fibonacci multinomial coefficients.

3. THE RELATION

Theorem: The recurrence relation for the Fibonacci multinomial coefficients is given by

$$\left\{ {n \atop s_1, \ldots, s_r} \right\}_u = \sum_{j=1}^{r} \left\{ {n-1 \atop s_1 - \delta_{1j}, \ldots, s_r - \delta_{rj}} \right\}_u U_{r-j+1,2+m}^{(r)}$$

in which $s_i = n - m - i + 1$

and $\quad m = n(1 - 1/r) + \frac{1}{2}(1 - r).$

Proof: We note first that

$$\sum_{i=1}^{r} s_i = \sum_{i=1}^{r} (n - m - i + 1) = r(n - m + 1) - \sum_{i=1}^{r} i$$

$$= rn - rm + r - \frac{1}{2}r(r + 1)$$

$$= rn + \frac{1}{2}r - \frac{1}{2}r^2 - rm = n.$$

Using the lemma, we have that

$$\sum_{j=1}^{r} \left\{ {n-1 \atop s_1 - \delta_{1j}, \ldots, s_r - \delta_{rj}} \right\}_n U_{r-j+1,r+m}^{(r)}$$

$$= \frac{u_{n-1}^{(r)}!\left\{u_{s_1}^{(r)} U_{r,r+m}^{(r)} + \cdots + u_{s_r}^{(r)} U_{1,r+m}^{(r)}\right\}}{u_{s_1}^{(r)}! \ldots u_{s_r}^{(r)}!}$$

$$= \frac{u_{n-1}^{(r)}!\left\{u_{n-m}^{(r)} U_{r,r+m}^{(r)} + \cdots + u_{n-m-r+1}^{(r)} U_{1,r+m}^{(r)}\right\}}{u_{s_1}^{(r)}! \ldots u_{s_r}^{(r)}!}$$

$$= u_{n-1}^{(r)}!u_n^{(r)} / u_{s_1}^{(r)}! \ldots u_{s_2}^{(r)}! \text{ (from the lemma)}$$

$$= \left\{ {n \atop s_1, \ldots, s_r} \right\}_u \text{ as required.}$$

4. CONCLUSION

As an example, suppose $r = 2$, $n = 2k + 1$; then $m = k$, $s_1 = k + 1$, and $s_2 = k$, and the theorem becomes

$$\left\{ {2k+1 \atop k} \right\}_u = \frac{u_{2k}^{(2)}!U_{2,k+2}^{(2)}}{u_k^{(2)}!u_k^{(2)}!} + \frac{u_{2k}^{(2)}!U_{1,k+2}^{(2)}}{u_{k+1}^{(2)}!u_{k-1}^{(2)}!}$$

$$= U_{2,k+2}^{(2)}\left\{ {2k \atop k} \right\}_u + U_{1,k+2}^{(2)}\left\{ {2k \atop k-1} \right\}_u.$$

This is the same as the equivalent result (F) in Hoggatt [3] (in our notation):

$$\left\{ {2k+1 \atop k} \right\}_u = U_{2,k+2}^{(2)}\left\{ {2k \atop k} \right\}_u - P_{22} U_{2,k+1}^{(2)}\left\{ {2k \atop k-1} \right\}_u$$

as it can be readily shown that

$$U_{1,k+2}^{(2)} = -P_{22}U_{2,k+1}^{(2)}.$$

The first five values of $U_{j,n}^{(2)}$, $j = 1, 2$, are:

$U_{j,n}^{(2)}$	$n = 1$	2	3	4	5
$j = 1$	1	0	$-P_{22}$	$-P_{21}P_{22}$	$-P_{21}^2P_{22} + P_{22}^2$
2	0	1	P_{21}	$P_{21}^2 - P_{22}$	$P_{21}^3 - 2P_{21}P_{22}$

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GENERALIZED FIBONACCI NUMBERS AS ELEMENTS OF IDEALS

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Wyler [3] has looked at the structure of second-order recurrences by considering them as elements of a commutative ring with the Lucas recurrence as unit element.

It is possible to supplement Wyler's results and to gain further insight into the structure of recurrences by looking at ideals in this commutative ring.

The purpose of this note is to look briefly at the structure of Horadam's generalized sequence of numbers [2] defined recursively by

$$w_n = pw_{n-1} - qw_{n-2} \quad (n \geq 2) \tag{1}$$

with $w_0 = a$, $w_1 = b$, and where p,q are arbitrary integers.

DeCarli [1] has examined a similarly generalized sequence over an arbitrary ring. It is proposed here to assume that the sequence $\{w_n\}$ of numbers are elements of a commutative ring R and to examine $\{w_n\}$ in terms of ideals of R. To this end, suppose that p,q are elements of an ideal of R.

$\langle p \rangle, \langle q \rangle$ are then the ideals generated by p and q, respectively, and $(\langle p \rangle, \langle q \rangle)$ is the sum of the ideals generated by p and q.