To prove the stronger result that if

$$\{v_m\} \equiv \{w_n\} \text{ for any } n,m, \text{ then } \{v_m\} \equiv \{w_n\} \text{ for all } n,m,$$

it would be necessary to replace "small" with "large" in the enunciation of Theorem 3. This would require S to be a prime ideal which could be achieved by embedding S in a maximal ideal $\mathcal{M}\alpha\beta$ which could be proved prime. However, this would then require restrictions on p' and q' as it would be easy to show that $q'v_{N-1} \in S$ but it would not automatically follow that $v_{N-1} \in S$.

Another problem that might be worth investigating is to look for commutators for relations like

$$w_{n+1}^p - w_n^p - w_{n-1}^p, \text{ where } p \text{ is a prime.}$$

These could be useful in Lie algebras.

REFERENCES

1. D. J. DeCarli. "A Generalized Fibonacci Sequence over an Arbitrary Ring." *The Fibonacci Quarterly* 8, No. 2 (1970):182-184.
2. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* 3, No. 3 (1965):161-176.
3. O. Wyler. "On Second Order Recurrences." *American Mathematical Monthly* 72, No. 5 (1965):500-506.

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A GENERALIZATION OF HILTON'S PARTITION OF HORADAM'S SEQUENCES

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1. INTRODUCTION

If $P_{r1}, P_{r2}, \ldots, P_{rr}$ are distinct integers for positive r, let

$$\omega = \omega(P_{r1}, \ldots, P_{rr})$$

be the set of integer sequences

$$\left\{W_{sn}^{(r)}\right\} = \left\{W_{s0}^{(r)}, W_{s1}^{(r)}, W_{s2}^{(r)}, \ldots\right\}$$

which satisfy the recurrence relation of order r,

$$W_{s,n+r}^{(r)} = \sum_{j=1}^{r} (-1)^{j+1} P_{rj} W_{s,n+r-j}^{(r)}, \quad (s = 1, 2, \ldots, r), \quad n \geq 1. \tag{1.1}$$

This is a generalization of of $\left\{W_{sn}^{(2)}\right\}$ studied in detail by Horadam [1, 2, 3, 4, 5].

Hilton [6] partitioned Horadam's sequence into a set F of generalized Fibonacci sequences and a set L of generalized Lucas sequences. We extend this to show that ω can be partitioned naturally into r sets of generalized sequences.

2. NOTATION

We define r sequences of order r, $\left\{V_{sn}^{(r)}\right\}$ $(s = 1, 2, \ldots, r)$ by

$$V_{sn}^{(r)} = d^{1-s}\sum_{j=1}^{r} A_{sj}^{(r)}\, \alpha_{rj}^{n}, \quad n \geq 1, \tag{2.1}$$

where the α_{rj} are the distinct roots of

$$x^{r} = \sum_{k=1}^{r} (-1)^{k+1} P_{rk}\, x^{r-k} \tag{2.2}$$

and

$$d = \det D$$

where D is the Vandermonde matrix

$$D = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rr} \\ \alpha_{r1}^{2} & \alpha_{r2}^{2} & \cdots & \alpha_{rr}^{2} \\ \alpha_{r1}^{r-1} & \alpha_{r2}^{r-1} & \cdots & \alpha_{rr}^{r-1} \end{bmatrix}$$

and the $A_{sj}^{(r)}$ are suitable constants that depend on the initial values of the sequence:

$$\left\{V_{sn}^{(r)}\right\} \in \omega(P_{r1}, P_{r2}, \ldots, P_{rr}).$$

Proof:

$$\begin{aligned} V_{s,n+r}^{(r)} &= d^{1-s}\sum_{j=1}^{r} A_{sj}^{(r)}\, \alpha_{rj}^{n+r} \\ &= d^{1-s}\sum_{j=1}^{r} A_{sj}^{(r)}\, \alpha_{rj}^{n} \sum_{k=1}^{r} (-1)^{k+1} P_{rk}\, \alpha_{rj}^{r-k} \\ &= \sum_{k=1}^{r} (-1)^{k+1} P_{rk} \left(d^{1-s}\sum_{j=1}^{r} A_{sj}^{(r)}\, \alpha_{rj}^{n+r-k} \right) \\ &= \sum_{k=1}^{r} (-1)^{k+1} P_{rk}\, V_{s,n+r-k}^{(r)}, \text{ as required.} \end{aligned}$$

3. THE PARTITION OF $\omega(P_{r1}, \ldots, P_{rr})$

It follows from (2.1) that we can represent $V_{rn}^{(r)}$ by

$$V_{rn}^{(r)} = d^{t-r}\sum_{j=1}^{r} B_{tj}^{(r)}\, \alpha_{rj}^{n} \quad (t = 1, 2, \ldots, r)$$

so that

$$B_{1j}^{(r)} \equiv A_{rj}^{(r)},$$

and $V_{rn}^{(r)}$ can be put in the form of any of the $V_{sn}^{(r)}$. For example, when $t = 3$,

$$V_{rn}^{(r)} = d^{3-r}\sum_{j=1}^{r} B_{3j}^{(r)} \alpha_{rj}^{n}$$

has the form of

$$V_{r-2,n}^{(r)} = d^{3-r}\sum_{j=1}^{r} A_{r-2,j}^{(r)}\alpha_{rj}^{n}.$$

We shall now consider the derivation of one sequence from another, so that in what follows the results hold for any of the r sequences. Thus there are r such partitions.

We say that $W_{sn}^{(r)}$ is in Fibonacci form when it is represented as in

$$W_{sn}^{(r)} = \frac{1}{d}\sum_{j=1}^{r} A_{rj}\alpha_{rj}^{n} \qquad |d| \neq 1 \tag{3.1}$$

and in Lucas form when it is represented as in

$$W_{sn}^{(r)} = \sum_{j=1}^{r} B_{rj}\alpha_{rj}^{n} \qquad |d| \neq 1 \tag{3.2}$$

where the B_{rj} are different constants from the A_{rj}. This is analogous to Hilton. To continue the analogy, one can see from (2.1) that there are r such forms which correspond to the distinct values of s. When $W_{sn}^{(r)}$ is in Fibonacci form we may perform an operation (′) to obtain a number

$$W_{sn}^{(r)\prime}$$

where

$$W_{sn}^{(r)\prime} = \sum_{j=1}^{r} B_{rj}\alpha_{rj}^{n}.$$

We say (like Hilton) that the sequence $\left\{W_{sn}^{(r)\prime}\right\}$ is *derived from* the sequence $\left\{W_{sn}^{(r)}\right\}$. Throughout this paper we assume that $|d|$ *is not unity*, because when d is unity the essential distinction between (3.1) and (3.2) breaks down. There would still be r partitions, provided the $A_{sj}^{(r)}$ of Equation (2.1) are distinct for all values of s, but the groups of sequences would have the basic Lucas form. Now

$$W_{sn}^{(r)\prime} = \sum_{j=1}^{r} A_{rj}\alpha_{rj}^{n} = d\left(\frac{1}{d}\sum_{j=1}^{r} A_{rj}\alpha_{rj}^{n}\right) = dW_{sn}^{(r)},$$

and so $W_{sn}^{(r)\prime\prime} = d^2 W_{sn}^{(r)}$, which corresponds to Hilton's Theorem 1.

It follows from (3.1) and Jarden [7] that

$$D\underset{\sim}{a} = d\underset{\sim}{w}$$

where $\underset{\sim}{a} = [A_{r1}, A_{r2}, \ldots, A_{rr}]^T$

and $\underset{\sim}{w} = \left[W_{s0}^{(r)}, W_{s1}^{(r)}, \ldots, W_{s,r-1}^{(r)}\right]^T.$

So
$$\underset{\sim}{a} = dD^{-1}\underset{\sim}{w}$$
in which
$$dD^{-1} \equiv [d_{\rho\kappa}]$$
is the matrix with $d_{\rho\kappa}$ in row ρ and column κ, where
$$d_{\rho\kappa} = (-1)^{\rho+\kappa} \prod_{\substack{\rho \neq m,n \\ m>n}} (\alpha_{rm} - \alpha_{rn}) \sum_{m \neq \rho} \alpha_{rm_1}\alpha_{rm_2} \cdots \alpha_{rm_{r-\kappa}}.$$
For $r = 2$,
$$D = \begin{bmatrix} 1 & 1 \\ \alpha_{21} & \alpha_{22} \end{bmatrix}, \quad D^{-1} = \frac{1}{d}\begin{bmatrix} \alpha_{22} & -1 \\ -\alpha_{21} & 1 \end{bmatrix}$$
where $d = \alpha_{22} - \alpha_{21}$.

For $r = 3$,
$$D = \begin{bmatrix} 1 & 1 & 1 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \alpha_{31}^2 & \alpha_{32}^2 & \alpha_{33}^2 \end{bmatrix},$$
$$D^{-1} = \frac{1}{d}\begin{bmatrix} \alpha_{33}\alpha_{32}(\alpha_{33} - \alpha_{32}), & -(\alpha_{33} + \alpha_{32})(\alpha_{33} - \alpha_{32}), & (\alpha_{33} - \alpha_{32}) \\ & \text{etc.} & \end{bmatrix}$$
where $d = (\alpha_{32} - \alpha_{31})(\alpha_{33} - \alpha_{31})(\alpha_{33} - \alpha_{32})$.

For $r = 4$,
$$D = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha_{41} & & & \\ \alpha_{41}^2 & & \text{etc.} & \\ \alpha_{41}^3 & & & \end{bmatrix}, \quad D^{-1} = \frac{1}{d}\begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ & & & \\ & \text{etc.} & & \\ & & & \end{bmatrix}$$
where $d = (\alpha_{42} - \alpha_{41})(\alpha_{43} - \alpha_{41})(\alpha_{43} - \alpha_{42})(\alpha_{44} - \alpha_{41})(\alpha_{44} - \alpha_{42})(\alpha_{44} - \alpha_{43})$
$$d_{11} = \alpha_{42}\alpha_{43}\alpha_{44}(\alpha_{43} - \alpha_{42})(\alpha_{44} - \alpha_{42})(\alpha_{44} - \alpha_{43})$$
$$d_{12} = -(\alpha_{42}\alpha_{43} + \alpha_{43}\alpha_{44} + \alpha_{44}\alpha_{42})(\alpha_{43} - \alpha_{42})(\alpha_{44} - \alpha_{42})(\alpha_{44} - \alpha_{43})$$
$$d_{13} = (\alpha_{42} + \alpha_{43} + \alpha_{44})(\alpha_{43} - \alpha_{42})(\alpha_{44} - \alpha_{42})(\alpha_{44} - \alpha_{43})$$
$$d_{14} = -(\alpha_{43} - \alpha_{42})(\alpha_{44} - \alpha_{42})(\alpha_{44} - \alpha_{43}).$$
From $\underset{\sim}{a} = dD^{-1}\underset{\sim}{w}$, we have
$$A_{r\rho} = \sum_{\kappa=1}^{r} d_{\rho\kappa} W_{s,\kappa-1}^{(r)} \quad \text{and} \quad W_{sn}^{(r)\prime} = \sum_{\rho=1}^{r} A_{r\rho}\alpha_{r\rho}^{n},$$

so

$$W_{sn}^{(r)\prime} = \sum_{\rho=1}^{r} \sum_{\kappa=1}^{r} d_{\rho\kappa} \alpha_{r\rho}^{n} W_{s,\kappa-1}^{(r)} \tag{3.3}$$

which is effectively a generalization of Equations (2) and (3) of Hilton.

Suppose

$$\left\{X_{sn}^{(r)}\right\} \text{ and } \left\{Y_{sn}^{(r)}\right\} \varepsilon\ \omega(P_{r1}, \ldots, P_{rr})$$

and that

$$X_{sn}^{(r)\prime} = Y_{sn}^{(r)} \quad (n = 0, 1, 2, \ldots).$$

Since

$$d\underset{\sim}{x}^{(r)} = \underset{\sim}{y}^{(r)},$$

$$Y_{sn}^{(r)} = \sum_{\rho=1}^{r} \sum_{\kappa=1}^{r} d_{\rho\kappa} \alpha_{r\rho}^{n} X_{s,\kappa-1}^{(r)}, \quad \text{from (3.3)} \tag{3.4}$$

and

$$d^2 X_{sn}^{(r)} = dY_{sn}^{(r)}$$

$$= \sum_{\rho=1}^{r} \sum_{\kappa=1}^{r} d_{\rho\kappa} \alpha_{r\rho}^{n} X_{s,\kappa-1}^{(r)}, \text{ from (3.4).} \tag{3.5}$$

(3.5) is a generalization of Theorem 2(i) of Hilton.

The analogue of Theorem 2(ii) of Hilton can be stated as:

if $\left\{W_{sn}^{(r)}\right\} \varepsilon\ \omega(P_{r1}, \ldots, P_{rr})$, $d^2 \Big| \sum_{\rho} \sum_{\kappa} d_{\rho\kappa} \alpha_{r\rho}^{n} W_{s,\kappa-1}^{(r)}$, $n < r$,

then $\left\{W_{sn}^{(r)}\right\} = \left\{X_{sn}^{(r)\prime}\right\}$ for some $\left\{X_{sn}^{(r)}\right\} \varepsilon\ \omega(P_{r1}, \ldots, P_{rr})$.

Proof: If $\left\{X_{sn}^{(r)}\right\} \varepsilon\ \omega(P_{r1}, \ldots, P_{rr})$,

then $$X_{sn}^{(r)\prime} = \sum_{\rho} \sum_{\kappa} d_{\rho\kappa} \alpha_{r\rho}^{n} X_{s,\kappa-1}^{(r)}, \text{ from (3.3).}$$

If $$X_{sn}^{(r)} = d^{-2} \sum_{\rho} \sum_{\kappa} d_{\rho\kappa} \alpha_{r\rho}^{n} W_{s,\kappa-1}^{(r)}, \; n < r,$$

then $$X_{sn}^{(r)\prime} = \frac{1}{d} \sum_{\rho} \sum_{\kappa} d_{\rho\kappa} \alpha_{r\rho}^{n} W_{s,\kappa-1}^{(r)}, \quad n < r,$$

but $$W_{sn}^{(r)} = \frac{1}{d} \sum_{\rho} \sum_{\kappa} d_{\rho\kappa} \alpha_{r\rho}^{n} W_{s,\kappa-1}^{(r)}, \quad n < r.$$

So $$dX_{sn}^{(r)} = W_{sn}^{(r)} \quad \text{for } n < r,$$

from which the result follows.

4. THEOREMS

The basic linear relationships between $\left\{X_{sn}^{(r)}\right\}$ and $\left\{Y_{sn}^{(r)}\right\}$ are described in the following theorem.

Theorem A: The following are equivalent:

$$\left\{X_{sn}^{(r)\prime}\right\} = \left\{Y_{sn}^{(r)}\right\} \tag{4.1}$$

$$Y_{s,n+m}^{(r)} = \sum_{\rho=1}^{r}\sum_{\kappa=1}^{r} d_{\rho\kappa}\alpha_{r\rho}^{m} X_{s,n+\kappa-1}^{(r)}, \text{ for all } n \geq 0, \tag{4.2}$$

$$X_{s,n+m}^{(r)} = \frac{1}{d^2}\sum_{\rho=1}^{r}\sum_{\kappa=1}^{r} d_{\rho\kappa}\alpha_{r\rho}^{m} Y_{s,n+\kappa-1}^{(r)}, \text{ for all } n \geq 0. \tag{4.3}$$

Proof: For each of (4.2) and (4.3) we need only require that the expression is true for r adjacent values of n.

(4.1) $\Longrightarrow$ (4.2);

if
$$\left\{X_{sn}^{(r)\prime}\right\} = \left\{Y_{sn}^{(r)}\right\},$$

then
$$Y_{s,n}^{(r)} = \sum_{\rho=1}^{r}\sum_{\kappa=1}^{r} d_{\rho\kappa}\alpha_{r\rho}^{n} X_{s,\kappa-1}^{(r)}, \text{ from (3.3).}$$

Thus (4.2) is true for $n=0$. Let $t \geq r$ and assume (4.2) is true for $0 \leq n < t$.

$$\begin{aligned}
Y_{s,t+m}^{(r)} &= \sum_{j=1}^{r}(-1)^{j+1}P_{rj}\,Y_{s,t+m-j}^{(r)}, \text{ from (1.1)},\\
&= \sum_{j=1}^{r}\sum_{\rho=1}^{r}\sum_{\kappa=1}^{r} d_{\rho\kappa}\alpha_{r\rho}^{m}(-1)^{j+1}P_{rj}\,X_{s,t-j+\kappa-1}^{(r)}\\
&= \sum_{\rho=1}^{r}\sum_{\kappa=1}^{r} d_{\rho\kappa}\alpha_{r\rho}^{m}\sum_{j=1}^{r}(-1)^{j+1}P_{rj}\,X_{s,t-j+\kappa-1}^{(r)}\\
&= \sum_{\rho}\sum_{\kappa} d_{\rho\kappa}\alpha_{r\rho}^{n} X_{s,t+\kappa-1}^{(r)}, \text{ as required.}
\end{aligned}$$

Similarly, (4.3) follows if we use (3.3) and induction.

(4.3) $\Longrightarrow$ (4.1);

since
$$X_{sn}^{(r)} = \frac{1}{d^2}\sum_{\rho}\sum_{\kappa} d_{\rho\kappa}\alpha_{r\rho}^{n} Y_{s,\kappa-1}^{(r)}, \text{ for } n < r,$$

it follows from the generalization of Hilton's Theorem 2(ii) that
$$\left\{X_{sn}^{(r)\prime}\right\} = \left\{Y_{sn}^{(r)}\right\}.$$

Similarly, it can be shown that (4.2) $\Longrightarrow$ (4.1). This completes the proof of Theorem A.

We now describe a partition of $\omega(P_{r1}, \ldots, P_{rr})$. If

$$\left\{W_{sn}^{(r)}\right\} \in \omega(P_{r1}, \ldots, P_{rr}),$$

let $W_{sn}^{(r)} = d^2 \omega_{sn}^{(r)}$ for all $n \geq 0$ where $m \geq 0$ is an integer,

$$\left\{\omega_{sn}^{(r)}\right\} \in \omega \quad \text{and} \quad d^2 \nmid \omega_{sn}^{(r)}$$

for at least one $n \geq 0$. Then, for $|d_i| \neq 1$,

$$\left\{W_{sn}^{(r)}\right\} \in L,$$

if $d^2 \mid \sum_{\rho} \sum_{\kappa} d_{\rho\kappa} \alpha_{r\rho}^n \omega_{s,\kappa-1}^{(r)}$, for all n, $0 \leq n < r$;

$$\left\{W_{sn}^{(r)}\right\} \in F,$$

if $d^2 \nmid \sum_{\rho} \sum_{\kappa} d_{\rho\kappa} \alpha_{r\rho}^n \omega_{s,\kappa-1}^{(r)}$, for at least one n, $0 \leq n < r$.

In view of Theorem A, if $\left\{W_{sn}^{(r)}\right\}$ is a member of F (or L), then any "tail" of $\left\{W_{sn}^{(r)}\right\}$ is also a member of F (or L), respectively. Note that this partition of $\left\{W_{sn}^{(r)}\right\}$ is not unique, since in terms of (2.1) L corresponds to $s = 1$ and F corresponds to $s = 2$. We could proceed with similar partitions for $s = 3, \ldots, r$, but they do not tell us anything essentially new.

<u>*Theorem B*</u>: $\left\{X_{sn}^{(r)}\right\} \in F$ iff $\left\{Y_{sn}^{(r)}\right\} \in L$.

<u>*Proof*</u>: (i) If $\left\{X_{sn}^{(r)}\right\} \in F$, suppose that

$$X_{sn}^{(r)} = d^{2m} x_{sn}^{(r)} \quad \text{for all } n \geq 0,$$

where $m \geq 0$ is an integer, and

$$\left\{x_{sn}^{(r)}\right\} \in F \quad \text{and} \quad d^2 \mid x_{sn}^{(r)} \quad \text{for at least one } n \geq 0.$$

Clearly $d^2 \nmid x_{s0}^{(r)}$, or $d^2 \nmid x_{s1}^{(r)}$, ..., or $d^2 \nmid x_{s,r-1}^{(r)}$. By Theorem A,

$$Y_{sn}^{(r)} = \sum_{\rho} \sum_{\kappa} d_{\rho\kappa} \alpha_{r\rho}^n X_{s,\kappa-1}^{(r)} \quad \text{for } 0 \leq n < r.$$

Let $$Y_{sn}^{(r)} = d^{2m} y_{sn}^{(r)} \quad \text{for all } n \geq 0.$$

Then $$y_{sn}^{(r)} = \sum_{\rho} \sum_{\kappa} d_{\rho\kappa} \alpha_{r\rho}^n x_{s,\kappa-1}^{(r)}, \quad 0 \leq n < r.$$

Since $x_{sn}^{(r)} \in F$, $d^2 \nmid \sum_{\rho} \sum_{\kappa} d_{\rho\kappa} \alpha_{r\rho}^n x_{s,\kappa-1}^{(r)}$ for at least one n, $0 \leq n < r$.

Therefore, $d^2 \nmid y_{sn}^{(r)}$ for at least one n, $0 \leq n < r$.

But it follows from Theorem A that for all n, $0 \le n < r$,

$$d^2 x_{sn}^{(r)} = \sum_{\rho} \sum_{\kappa} d_{\rho\kappa} \alpha_{r\rho}^n y_{s,\kappa-1}^{(r)}.$$

Therefore $\left\{y_{sn}^{(r)}\right\} \varepsilon L$, and so $\left\{y_{sn}^{(r)}\right\} \varepsilon L$.

(ii) If $\left\{Y_{sn}^{(r)}\right\} \varepsilon L$, suppose that

$$Y_{sn}^{(r)} = d^{2m} y_{sn}^{(r)} \quad \text{for all } n \ge 0,$$

where $m \ge 0$ is an integer, and

$$\left\{y_{sn}^{(r)}\right\} \varepsilon L \quad \text{and} \quad d^2 \nmid y_{sn}^{(r)} \text{ for at least one } n \ge 0.$$

Clearly $d^2 \nmid y_{s0}^{(r)}$, or $d^2 \nmid y_{s1}^{(r)}$, ..., or $d^2 \nmid y_{s,r-1}^{(r)}$. By Theorem A,

$$X_{sn}^{(r)} = \frac{1}{d^2} \sum_{\rho} \sum_{\kappa} d_{\rho\kappa} \alpha_{r\rho}^n y_{s,\kappa-1}^{(r)} \text{ for } 0 \le n < r.$$

Let $$X_{sn}^{(r)} = d^{2m} x_{sn}^{(r)} \text{ for all } n \ge 0.$$

Then $$x_{sn}^{(r)} = \frac{1}{d^2} \sum_{\rho} \sum_{\kappa} d_{\rho\kappa} \alpha_{r\rho}^n y_{s,\kappa-1}^{(r)}, \quad 0 \le n < r.$$

Since $\left\{y_{sn}^{(r)}\right\} \varepsilon L$, $d^2 \,\Big|\, \sum_{\rho} \sum_{\kappa} d_{\rho\kappa} \alpha_{r\rho}^n y_{s,\kappa-1}^{(r)}$ for all n, $0 \le n < r$.

So $x_{s0}^{(r)}, x_{s1}^{(r)}, \ldots, x_{s,r-1}^{(r)}$ are integers and so $\left\{x_{sn}^{(r)}\right\} \varepsilon \omega$. But

$$y_{sn}^{(r)} = \sum_{\rho} \sum_{\kappa} d_{\rho\kappa} \alpha_{r\rho}^n x_{s,\kappa-1}^{(r)} \text{ for all } n, \ 0 \le n < r,$$

and since $d^2 \nmid y_{s0}^{(r)}$, or $d^2 \nmid y_{s1}^{(r)}$, ..., or $d^2 \nmid y_{s,r-1}^{(r)}$, it follows that

$$d^2 \nmid \sum_{\rho} \sum_{\kappa} d_{\rho\kappa} \alpha_{r\rho}^n x_{s,\kappa-1}^{(r)} \text{ for al least one } n, \ 0 \le n < r.$$

Therefore $\left\{x_{sn}^{(r)}\right\} \varepsilon F$, and so $\left\{X_{sn}^{(r)}\right\} \varepsilon F$. This completes the proof of Theorem B.

At this point, Hilton considered identities obtained from the binomial theorem. The corresponding application of the multinomial theorem to the roots α_{rj} of the auxiliary equation seems too complicated to pursue, though it is possible.

Another approach is to modify the method of Williams [10]: let

$$\varepsilon = \exp(2i\pi/r), \text{ where } i^2 = -1,$$

and as before

$$d = \prod_{\substack{j,k=1 \\ j>k}}^{r} (\alpha_{rj} - \alpha_{rk}).$$

If we let

$$\alpha_{rj} = \frac{1}{r}\sum_{k=0}^{r-1} W_{k,r+1}^{(r)} d^k \varepsilon^{-jk} \quad (j = 1, 2, \ldots, r),$$

then it is shown in Shannon [9] that

$$\alpha_{rj}^m = r^{-1}\sum_{k=0}^{r-1} W_{k,n+r}^{(r)} d^k \varepsilon^{-jk},$$

which seems to be a more useful form than the corresponding multinomial expression.

REFERENCES

1. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* 3 (1965):161-176.
2. A. F. Horadam. "Generating Functions for Powers of a Certain Generalized Sequence of Numbers." *Duke Math. J.* 32 (1965):437-446.
3. A. F. Horadam. "Generalizations of Two Theorems of K. Subba Rao." *Bull. Calcutta Math. Soc.* 58, Nos. 1 & 2 (1966):23-29.
4. A. F. Horadam. "Special Properties of the Sequence $W(a,b;p,q)$." *The Fibonacci Quarterly* 5 (1967):424-434.
5. A. F. Horadam. "Tschebyscheff and Other Functions Associated with the Sequence $\{W_n(a,b;p,q)\}$." *The Fibonacci Quarterly* 7 (1969):14-22.
6. A. J. W. Hilton. "On the Partition of Horadam's Generalized Sequences into Generalized Fibonacci and Generalized Lucas Sequences." *The Fibonacci Quarterly* 12 (1974):339-345.
7. Dov Jarden. *Recurring Sequences.* Jerusalem: Riveon Lematematika, 1966, p. 107.
8. Edouard Lucas. "Théorie des fonctions numériques périodiques." *American J. Math.* 1 (1878):184-240, 289-321.
9. A. G. Shannon. "A Generalization of the Hilton-Ferns Theorem on the Expansion of Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* 12 (1974):237-240.
10. H. C. Williams. "On a Generalization of the Lucas Functions." *Acta Arithmetica* 20 (1972):33-51.

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