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# SOME RESTRICTED MULTIPLE SUMS

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## 1. INTRODUCTION

Let $a$, $b$ be positive integers, $(a,b) = 1$. Consider the sum

$$S = \sum_{br+as<ab} x^{br+as} \equiv \sum_{\substack{r=0 \\ br+as<ab}}^{a-1} \sum_{s=0}^{b-1} x^{br+as}. \tag{1.1}$$

We will show that

$$S = \frac{1 - x^{ab}}{(1 - x^a)(1 - x^b)} - \frac{x^{ab}}{1 - x}. \tag{1.2}$$

As an application of (1.2), let $B_n(x)$ denote the Bernoulli polynomial of degree $n$ defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}, \quad B_n = B_n(0).$$

Then we have

$$\sum_{br+as<ab} B_n\left(x + \frac{r}{a} + \frac{s}{b}\right) = (B(ab) + abB(x))^n - (aB + bB(ax))^n, \tag{1.3}$$

where

$$(uB(x) + vB(y))^n \equiv \sum_{k=0}^{n} \binom{n}{k} u^k v^{n-k} B_k(x) B_{n-k}(y).$$

We also evaluate the sum

$$\sum_{br+as<ab} (x + br + as)^n \tag{1.4}$$

in terms of Bernoulli polynomials; see (3.8) below.

Let $a$, $b$, $c$ be positive integers such that $(b,c) = (c,a) = (a,b) = 1$. The sum (1.1) suggests the consideration of the sums

$$S_1 = \sum_{bcr+cas+abt<abc} x^{bcr+cas+abt}$$

and

$$S_2 = \sum_{bcr+cas+abt<2abc} x^{bcr+cas+abt}$$

where $0 \le r < a$, $0 \le s < b$, $0 \le t < c$. We are unable to evaluate $S_1$ and $S_2$ separately. However, we show that

(1.5) $$x^{abc}S_1 + S_2 = \frac{(1 - x^{abc})^2}{(1 - x^{bc})(1 - x^{ca})(1 - x^{ab})} - \frac{x^{2abc}}{1 - x}.$$

For applications to triple sums analogous to (1.3) and (1.4) see (5.5), (5.6), and (5.7) below.

We remark that the case $x = 0$ of (1.3) is implicit in the proof of Theorem 1 of [1].

## 2. PROOF OF (1.2)

We have

$$S = \sum_{br+as<ab} x^{br+as} = \sum_{r=0}^{a-1} x^{br} \sum_{as<b(a-r)} x^{as} = \sum_{r=0}^{a-1} x^{br} \sum_{s=0}^{[b(a-r)/a]} x^{as}.$$

Since

$$[b(a - r)/a] = b - [br/a] - 1,$$

it follows that

(2.1) $$S = \sum_{r=0}^{a-1} x^{br}\frac{1 - x^{ab-a[br/a]}}{1 - x^a} = \frac{1 - x^{ab}}{(1 - x^a)(1 - x^b)} - \frac{x^{ab}}{1 - x^a}\sum_{r=0}^{a-1} x^{br-a[br/a]}.$$

Clearly the exponent

(2.2) $$br - a[br/a] \qquad (r = 0, 1, \ldots, a - 1)$$

is the remainder obtained in dividing $br$ by $a$. Since $(a,b) = 1$, it follows that the numbers (2.2) are a permutation of $0, 1, \ldots, a - 1$. Hence, (2.1) becomes

$$S = \frac{1 - x^{ab}}{(1 - x^a)(1 - x^b)} - \frac{x^{ab}}{1 - x^a}\frac{1 - x^a}{1 - x},$$

so that

(2.3) $$S = \frac{1 - x^{ab}}{(1 - x^a)(1 - x^b)} - \frac{x^{ab}}{1 - x}.$$

This proves (1.2). Note that the complementary sum

(2.4) $$\overline{S} = \sum_{\substack{r=0 \\ br+as>ab}}^{a-1} \sum_{s=0}^{b-1} x^{br+as}$$

satisfies

$$S + \overline{S} = \frac{1 - x^{ab}}{1 - x^a}\frac{1 - x^{ab}}{1 - x^b}.$$

Hence, by (2.3),

(2.5) $$\overline{S} = \frac{x^{ab}}{1 - x} - \frac{x^{ab}(1 - x^{ab})}{(1 - x^a)(1 - x^b)}.$$

## 3. SOME APPLICATIONS

In (1.1) and (1.2), replace $x$ by $e^{z/ab}$:

(3.1) $$\sum_{br+as<ab} e^{(br+as)z/ab} = \frac{e^z}{e^{z/ab} - 1} - \frac{e^z - 1}{(e^{z/a} - 1)(e^{z/b} - 1)}.$$

Multiplying by $z^2e^{xz}/(e^z - 1)$, we get

$$\text{(3.2)}\qquad \frac{z^2}{e^z - 1}\sum_{br+as<ab} e^{(br+as)z/ab+xz} = \frac{z^2e^{(x+1)z}}{(e^z - 1)(e^{z/ab} - 1)} - \frac{z^2e^{xz}}{(e^{z/a} - 1)(e^{z/b} - 1)}.$$

Since

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!},$$

(3.2) becomes

$$z\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{br+as<ab} B_n\left(x + \frac{r}{a} + \frac{s}{b}\right) = \frac{z^2e^{(x+1)z}}{(e^z - 1)(e^{z/ab} - 1)} - \frac{z^2e^{xz}}{(e^{z/a} - 1)(e^{z/b} - 1)}$$

$$\text{(3.3)}\qquad = ab\sum_{j,k=0}^{\infty} B_j(x)B_k(ab)\frac{z^j(z/ab)^k}{j!k!} - ab\sum_{j,k=0}^{\infty} B_j(ax)B_k\frac{(z/a)^j(a/b)^k}{j!k!}.$$

Equating coefficients of $z^n$, we get

$$\text{(3.4)}\qquad n\sum_{br+as<ab} B_{n-1}\left(x + \frac{r}{a} + \frac{s}{b}\right) = (ab)^{1-n}\sum_{k=0}^{n}\binom{n}{k}(ab)^{n-k}B_{n-k}(x)B_k(ab) - (ab)^{1-n}\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}B_{n-k}(ax)B_k.$$

This can be written more compactly in the form

$$\text{(3.5)}\qquad n(ab)^{n-1}\sum_{br+as<ab} B_{n-1}\left(x + \frac{r}{a} + \frac{s}{b}\right) = (B(ab) + abB(x))^n - (aB + bB(ax))^n,$$

where it is understood that

$$(uB(x) + vB(y))^n = \sum_{k=0}^{n}\binom{n}{k}u^kv^{n-k}B_k(x)B_{n-k}(y).$$

Alternatively, (3.3) can be replaced by

$$ab\sum_{j,k=0}^{\infty} B_j(1)B_k(abx)\frac{z^j(z/ab)^k}{j!k!} - ab\sum_{j,k=0}^{\infty} B_j(ax)B_k\frac{(z/a)^j(z/b)^k}{j!k!}.$$

Hence, we now get

$$\text{(3.6)}\qquad n(ab)^{n-1}\sum_{br+as<ab} B_{n-1}\left(x + \frac{r}{a} + \frac{s}{b}\right) = (B(abx) + abB(1))^n - (aB + bB(ax))^n.$$

Note that comparison of (3.6) with (3.5) gives

$$(3.7)\qquad (B(ab) + abB(x))^n = (B(abx) + abB(1))^n,$$

which indeed holds for arbitrary $a$, $b$. We also recall that

$$B_n(1) = B_n,\ (n \neq 1);\ B_1(1) = B_1 + 1 = \frac{1}{2}.$$

For $n = 1$, since $B_1 = -\frac{1}{2}$ and

$$\sum_{br+as<ab} 1 = \sum_{r=0}^{a-1} \sum_{s<\frac{b}{a}(a-r)} 1 = \sum_{r=0}^{a-1}\left(b - \left[\frac{br}{a}\right]\right)$$

$$= ab - \frac{1}{2}(a-1)(b-1),\ ((a,b) = 1),$$

(3.5) reduces to

$$ab - \frac{1}{2}(a-1)(b-1) = (B(ab) + abB(x))' - (aB + bB(ax))'$$

$$= \left(ab - \frac{1}{2}\right) + ab\left(x - \frac{1}{2}\right) + \frac{1}{2}a - b\left(ax - \frac{1}{2}\right),$$

which is correct.

In place of (3.2) we now take

$$z^2 \sum_{br+as<ab} e^{(br+as)z+xz} = \frac{z^2 e^{(x+ab)z}}{e^z - 1} - \frac{z^2(e^{abz} - 1)e^{xz}}{(e^{az} - 1)(e^{bz} - 1)}$$

$$= z\sum_{n=0}^{\infty} B_n(x + ab)\frac{z^n}{n!} - (ab)^{-1}\sum_{j,k=0}^{\infty}(B_j(b) - B_j)B_k\left(\frac{x}{b}\right)\frac{(az)^j(bz)^k}{j!k!}.$$

It follows that

$$(3.8)\qquad n(n-1)ab\sum_{br+as<ab}(x + br + as)^{n-2}$$

$$= nabB_{n-1}(x + ab) - \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k(B_{n-k}(b) - B_{n-k})B_k\left(\frac{x}{b}\right).$$

For example, for $n = 2$, since $B_2(x) = x^2 - x + \frac{1}{6}$, we have

$$2ab\left(ab - \frac{1}{2}(a-1)(b-1)\right) = 2ab\left(x + ab - \frac{1}{2}\right) - a^2(b^2 - b) - 2ab^2\left(\frac{x}{b} - \frac{1}{2}\right),$$

which is correct.

Note that, for $b = 1$, (3.8) becomes

$$n(n-1)a\sum_{r=0}^{a-1}(x + r)^{n-2} = naB_{n-1}(x + a) - \sum_{k=0}^{n}\binom{n}{k}a^{n-k}(B_{n-k}(1) - B_{n-k})B_k(x).$$

Since

$$B_n(1) = B_n,\ (n \neq 1);\ B_1(1) = \frac{1}{2},\ B_1 = -\frac{1}{2},$$

we get

$$n(n-1)a\sum_{r=0}^{a-1}(x + a)^{n-2} = na\left(B_{n-1}(x + a) - B_{n-1}(a)\right),$$

that is, the familiar formula (replacing $n - 1$ by $n$)

$$\sum_{r=0}^{a-1} (x + a)^{n-1} = \frac{1}{n}\{B_n(x + a) - B_n(x)\}.$$

Similarly, for $b = 1$, (3.5) reduces to

$$na^{n-1}\sum_{r=0}^{a-1} B_{n-1}\left(x + \frac{r}{a}\right) = (B(a) + aB(x))^n - (ab + B(ax))^n. \tag{3.9}$$

We recall [2, p. 21] that

$$B_{n-1}(ax) = a^{n-2}\sum_{r=0}^{a-1} B_{n-1}\left(x + \frac{r}{a}\right). \tag{3.10}$$

Comparison of (3.10) with (3.9) yields

$$(B(a) + aB(x))^n - (aB + B(ax))^n = naB_{n-1}(ax). \tag{3.11}$$

To give a direct proof of (3.11), let $R_n$ denote the left-hand side of (3.11). Then,

$$\begin{aligned}\sum_{n=0}^{\infty} R_n\frac{z^n}{n!} &= \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{n}\binom{n}{k} a^k B_k(x)B_{n-k}(a) - \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{n}\binom{n}{k} a^k B_k B_{n-k}(ax) \\ &= \frac{aze^{axz}}{e^{az}-1}\,\frac{ze^{az}}{e^z - 1} - \frac{az}{e^{az} - 1}\,\frac{ze^{axz}}{e^z - 1} \\ &= \frac{az^2e^{axz}}{e^z - 1} = az\sum_{n=0}^{\infty} B_n(ax)\frac{z^n}{n!},\end{aligned}$$

and (3.11) follows at once.

## 4. PROOF OF (1.5)

Put

$$S_1 = \sum_{bcr + cas + abt < abc} x^{bcr + cas + abt} \tag{4.1}$$

and

$$S_2 = \sum_{bcr + cas + abt < 2abc} x^{bcr + cas + abt}. \tag{4.2}$$

It is understood that in all such sums

$$0 \le r < a,\ 0 \le s < b,\ 0 \le t < c. \tag{4.3}$$

As for $S_1$, we have

$$\begin{aligned} S_1 &= \sum_{br + as < ab} x^{c(br+as)} \sum_{t < c - \frac{c}{ab}(br+as)} x^{abt} \\ &= \sum_{br + as < ab} x^{c(br+as)}\,\frac{1 - x^{ab(c - [c(br+as)/ab])}}{1 - x^{ab}} \\ &= \frac{1}{1 - x^{ab}}\sum_{br + as < ab} x^{c(br+as)} - \frac{x^{ab}}{1 - x^{ab}}\sum_{br + as < ab} x^{R(c(br+ab)/ab)} \end{aligned} \tag{4.4}$$

where $R(m/ab)$ denotes the remainder obtained in dividing $m$ by $ab$. It will be convenient to put

$$U = \{u \mid u = c(br + as),\ br + as < ab\}$$
$$V = \{v \mid v = c(br + as),\ br + as > ab\}.$$

Thus (4.4) becomes

$$S_1 = \frac{1}{1 - x^{ab}} \sum_{u \in U} x^u - \frac{x^{ab}}{1 - x^{ab}} \sum_{u \in U} x^{R(u/ab)}. \tag{4.5}$$

Next put

$$S_2' = \sum_{\substack{bcr + cas + abt < 2abc \\ br + as > ab}} x^{c(br + as) + abt}$$

$$S_2'' = \sum_{\substack{bcr + cas + abt < 2abc \\ br + as > ab}} x^{c(br + as) + abt},$$

so that $S_2 = S_2' + S_2''$. Clearly

$$S_2' = \sum_{br + as < ab} x^{c(br + as)} \sum_{t=0}^{c-1} x^{abt}$$

$$= \frac{1 - x^{abc}}{1 - x^{ab}} \sum_{u \in U} x^u \tag{4.6}$$

The evaluation of $S_2''$ is less simple. We have

$$S_2'' = \sum_{br + as > ab} x^{c(br + as)} \sum_{t < 2c - \frac{c}{ab}(br + as)} x^{abt} = \sum_{v \in V} x \sum_{t < 2c - (v/ab)} x^{abt}$$

$$= \sum_{v \in V} \frac{x^v 1 - x^{ab(2c - [v/ab])}}{1 - x^{ab}}$$

$$= \frac{1}{1 - x^{ab}} \sum_{v \in V} x^v - \frac{x^{2abc}}{1 - x^{ab}} \sum_{v \in V} x^{r(v/ab)}. \tag{4.7}$$

It follows from (4.5) and (4.7) that

$$x^{abc} S_1 + S_2'' = \frac{x^{abc}}{1 - x^{ab}} \sum_{u \in U} x^u + \frac{1}{1 - x^{ab}} \sum_{v \in V} x^v$$
$$- \frac{x^{2abc}}{1 - x^{ab}} \left\{ \sum_{u \in U} x^{R(u/ab)} + \sum_{v \in V} x^{R(v/ab)} \right\}.$$

Since

$$\sum_{u \in U} x^{R(u/ab)} + \sum_{v \in V} x^{R(v/ab)} = \sum_{t=0}^{ab-1} x^t = \frac{1 - x^{ab}}{1 - x},$$

we get

$$x^{abc}S_1 + S_2'' = \frac{x^{abc}}{1 - x^{ab}}\sum_{u \in U} x^u + \frac{1}{1 - x^{ab}} \sum_{v \in V} x^v - \frac{x^{2abc}}{1 - x}. \tag{4.8}$$

Hence, by (4.6) and (4.8), we have

$$\begin{aligned} x^{abc}S_1 + S_2' + S_2'' &= \left(\frac{x^{abc}}{1 - x^{ab}} + \frac{1 - x^{abc}}{1 - x^{ab}}\right)\sum_{u \in U} x^u + \frac{1}{1 - x^{ab}}\sum_{v \in V} x^v - \frac{x^{2abc}}{1 - x} \\ &= \frac{1}{1 - x^{ab}}\left\{\sum_{u \in U} x^u + \sum_{v \in V} x^v\right\} - \frac{x^{2abc}}{1 - x} \\ &= \frac{1}{1 - x^{ab}}\sum_{r=0}^{a-1}\sum_{s=0}^{b-1} x^{c(br+as)} - \frac{x^{2abc}}{1 - x} \\ &= \frac{1}{1 - x^{ab}}\,\frac{1 - x^{abc}}{1 - x^{bc}}\,\frac{1 - x^{abc}}{1 - x^{ac}} - \frac{x^{2abc}}{1 - x}. \end{aligned}$$

Therefore,

$$x^{abc}S_1 + S_2 = \frac{(1 - x^{abc})^2}{(1 - x^{bc})(1 - x^{ca})(1 - x^{ab})} - \frac{x^{2abc}}{1 - x}. \tag{4.9}$$

## 5. SOME RESTRICTED TRIPLE SUMS

It follows from (4.9) with $x$ replaced by $e^{z/abc}$ that

$$e^z\sum_{\sigma < 1} e^{\sigma z} + \sum_{\sigma < 2} e^{\sigma z} = \frac{(1 - e^z)^2}{(1 - e^{z/a})(1 - e^{z/b})(1 - e^{z/c})} - \frac{e^{2z}}{1 - e^{z/abc}}, \tag{5.1}$$

where for brevity we put

$$\sigma = \frac{r}{a} + \frac{s}{b} + \frac{t}{c}. \tag{5.2}$$

Multiplying both sides of (5.1) by $z^3e^{xz}/(e^z - 1)^2$, we get

$$\begin{aligned} &\frac{z^3}{(e^z - 1)^2}\sum_{\sigma < 1} e^{(x+\sigma+1)} + \frac{z^3}{(e^z - 1)^2}\sum_{\sigma < 2} e^{(x+\sigma)} \\ &= \frac{z^3e^{(x+2)z}}{(e^z - 1)^2(e^{z/abc} - 1)} - \frac{z^3e^{xz}}{(e^{z/a})(e^{z/b} - 1)(e^{z/c} - 1)}. \end{aligned} \tag{5.3}$$

In order to obtain a compact result we make use of Nörlund's definition of Bernoulli numbers of higher order [2, Chapter 6]. Let $\omega_1, \omega_2, \ldots, \omega_k$ denote parameters and define the polynomial $B_n(x|\omega_1, \ldots, \omega_k)$ by means of

$$\frac{\omega_1\omega_2 \ldots \omega_k z^k e^{xz}}{(e^{\omega_1 z} - 1)(e^{\omega_2 z} - 1) \ldots (e^{\omega_k z} - 1)} = \sum_{n=0}^{\infty} B_n^{(k)}(x|\omega_1, \ldots, \omega_k)\frac{z^n}{n!}. \tag{5.4}$$

With this notation, (5.3) becomes

$$z\sum_{n=0}^{\infty}\frac{z^n}{n!}\left\{\sum_{\sigma<1}B_n^{(2)}(x+\sigma+1|1,\ 1)+\sum_{\sigma<2}B_n^{(2)}(x+\sigma|1,\ 1)\right\}$$

$$=abc\sum_{n=0}^{\infty}\frac{z}{n!}\left\{B_n^{(3)}(x+2|1,\ 1,\ (abc)^{-1})-B_n^{(3)}(x|a^{-1},\ b^{-1},\ c^{-1})\right\}.$$

Hence, equating coefficients of $z$ , we get

$$n\left\{\sum_{\sigma<1}B_{n-1}^{(2)}(x+\sigma+1|1,\ 1)+\sum_{\sigma<2}B_{n-1}^{(2)}(x+\sigma|1,\ 1)\right\} \tag{5.5}$$

$$=abc\left\{B_n^{(3)}(x+2|1,\ 1,\ (abc)^{-1})-B_n^{(3)}(x|a^{-1},\ b^{-1},\ c^{-1})\right\}.$$

Similarly, it follows from (5.1) that

$$n(n-1)(n-2)\left\{\sum_{\sigma<1}(x+\sigma+1)^{n-3}+\sum_{\sigma<2}(x+\sigma)^{n-3}\right\} \tag{5.6}$$

$$=n(n-1)(abc)^{-n+1}B_{n-2}(abc(x+2))-\Delta_x^2B_n^{(3)}(x|a^{-1},\ b^{-1},\ c^{-1}),$$

where $\Delta_x^2$ is the familiar difference operator:

$$\Delta_x^2 f(x)=f(x+2)-2f(x+1)+f(x).$$

Finally, multiplying both sides of (5.1) by $z^3e^{xz}/(e^z-1)$, we get

$$n(n-1)\left\{\sum_{\sigma<1}B_{n-2}(x+\sigma+1)+\sum_{\sigma<2}B_{n-2}(x+\sigma)\right\} \tag{5.7}$$

$$=nabcB_{n-1}^{(2)}(x+2|1,\ (abc)^{-1})-nabc\Delta_xB_{n-1}^{(3)}(x|a^{-1},\ b^{-1},\ c^{-1}).$$

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