MULTISECTION OF THE FIBONACCI CONVOLUTION ARRAY AND GENERALIZED LUCAS SEQUENCE

VERNER E. HOGGATT, JR., AND MARJORIE BICKNELL-JOHNSON
San Jose State University, San Jose, CA 95192

1. INTRODUCTION

The general problem of multisecting a general sequence rapidly becomes very complicated. In this paper we multisect the convolutions of the Fibonacci sequence and certain generalized Lucas sequences.

When we m-sect a sequence, we write a generating function for every mth term of the sequence. To illustrate, we recall [1], [2],

$$\sum_{k=0}^{\infty} F_{mk+r}x^k = \frac{F_r + (-1)^r F_{m-r}x}{1 - L_m x + (-1)^m x^2}, \tag{1.1}$$

which m-sects the Fibonacci sequence $\{F_n\}$, where

$$F_1 = F_2 = 1,\ F_{n+1} = F_n + F_{n-1},$$

and where L_m is the mth term of the Lucas sequence $\{L_n\}$,

$$L_0 = 2,\ L_1 = 1,\ L_{n+1} = L_n + L_{n-1}.$$

For later comparison, it is well known that the Fibonacci and Lucas sequences enjoy the Binet forms

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n, \tag{1.2}$$

where α and β are the roots of $x^2 - x - 1 = 0$,

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Also, the generating functions for F_n and L_n are

$$\frac{x}{1-x-x^2} = \sum_{n=0}^{\infty} F_n x^n, \quad \frac{2-x}{1-x-x^2} = \sum_{n=0}^{\infty} L_n x^n. \tag{1.3}$$

The Fibonacci convolution array, written in rectangular form, is

1	1	1	1	1	...
1	2	3	4	5	...
2	5	9	14	20	...
3	10	22	40	65	...
5	20	51	105	190	...
8	38	111	256	511	...
...	...	...	...	...	...

where each column is the convolution of the succeeding column with the Fibonacci sequence. The convolution sequence $\{c_n\}$ of two sequences $\{a_n\}$ and $\{b_n\}$ is formed by

$$c_n = \sum_{k=1}^{n} a_k b_{n-k+1}.$$

Also, it is known that the generating functions of successive convolutions of the Fibonacce sequence are given by $(1 - x - x^2)^{-k-1}$, $k = 0, 1, 2, \ldots$, where $k = 0$ gives the Fibonacci sequence itself.

2. MULTISECTION OF THE FIBONACCI CONVOLUTION ARRAY

We now proceed to multisect the Fibonacci convolution array. Recalling (1.1), we let

$$G_r = \frac{F_r + (-1)\, F^r_{k-r}x}{1 - L_k x + (-1)\, x^2}, \qquad G_r^* = \frac{F_r + (-1)\, F^r_{k-r}x}{1 - L_k x^k + (-1)^k x^{2k}}.$$

Clearly,

$$\sum_{r=0}^{k-1} G_r^* x^r = \frac{1}{1 - x \quad x^2}.$$

Thus,

$$\sum_{r=0}^{k-1} (F_r + (-1)^r F_{k-r} x^k) x^r = Q_k(x),$$

where

$$Q_k(x) = \frac{1 - L_k x^k + (-1)^k x^{2k}}{1 - x - x^2}.$$

To multisect the general convolution sequence for the Fibonacci numbers, let us work on column s, where $s = 1$ is the Fibonacci sequence itself. Then

$$Q_k^s(x) = \left(\frac{1 - L_k x^k + (-1)^k x^{2k}}{1 - x - x^2}\right)^s.$$

Now there are k separate k-sectors. The coefficients of the numerator polynomial of the jth generator are given by every kth coefficient of $Q_k^s(x)$, beginning with $1 \le j \le k$, while the denominator is $(1 - L_k x^k + (-1)^k x^{2k})^s$.

It is now simple to see how to multisect the columns of Pascal's triangle (see [2]) by taking

$$Q^s(x) = \left(\frac{1 - x^k}{1 - x}\right)^s.$$

We can even multisect the negative powers, which in the Fibonacci case is just a finite polynomial $(1 - x - x^2)^s$ from which we take every kth coefficient.

3. THE TRIBONACCI AND HIGHER CONVOLUTION ARRAYS

Define the Tribonacci numbers $\{T_n\}$ by

(3.1) $$T_0 = 0,\ T_1 = T_2 = 1,\ T_{n+3} = T_{n+2} + T_{n+1} + T_n.$$

The Tribonacci convolution triangle, with the Tribonacci numbers appearing in the leftmost column, is

1	1	1	1	1	...
1	2	3	4	5	...
2	5	9	14	20	...
4	12	25	44	70	...
7	26	63	135	...	...

(continued)

13	56	153	...	...	...
24	...	...	...	...	...
...	...	...	...	...	...

Since

$$\frac{x}{1 - x - x^2 - x^3} = \sum_{n=0}^{\infty} T_n x^n, \tag{3.2}$$

the generating functions for the Tribonacci convolution sequences are given successively by

$$[x/(1 - x - x^2 - x^3)]^{k+1}, \; k = 0, 1, 2, \ldots,$$

where $k = 0$ gives the Tribonacci sequence itself.

Let

$$S_k = \alpha^k + \beta^k + \gamma^k$$

where α, β, and γ are the roots of $x^3 - x^2 - x - 1 = 0$. Then the multisecting generating functions are obtained from

$$Q_k(x) = \frac{1 - S_k x^k + S_{-k} x^{2k} - x^{3k}}{1 - x - x^2 - x^3}, \tag{3.3}$$

where the coefficients of $Q_k^s(x)$ used are

$$T_1, T_2, T_3, \ldots, T_k, (T_{k+1} - S_k), \ldots, (T_{k+s} - S_k T_s), T_{-k-1}, T_{-k}, \ldots, T_{-2}.$$

The coefficients of the numerator polynomial of the jth generator are given by every kth coefficient of $Q_k^s(x)$, beginning with $1 \le j \le k$, while the denominator is $(1 - S_k x^k + S_{-k} x^{2k} - x^{3k})^s$.

From the auxiliary polynomial $x^3 - x^2 - x - 1 = 0$,

$$T_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + \gamma T_{n-1} = \frac{\beta^n - \gamma^n}{\beta - \gamma} + \alpha T_{n-1} = \frac{\gamma^n - \alpha^n}{\gamma - \alpha} + \beta T_{n-1}$$

or

$$3T_n - T_{n-1} = \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\beta^n - \gamma^n}{\beta - \gamma} + \frac{\gamma^n - \alpha^n}{\gamma - \alpha}. \tag{3.4}$$

Also,

$$T_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + \gamma\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + \gamma^2\frac{\alpha^{n-2} - \beta^{n-2}}{\alpha - \beta} + \cdots + \gamma^n. \tag{3.5}$$

For the Quadranacci numbers $\{Q_n\}$ defined by

$$Q_0 = 0,\; Q_1 = Q_2 = 1,\; Q_3 = 2,\; Q_{n+4} = Q_{n+3} + Q_{n+2} + Q_{n+1} + Q_n \tag{3.6}$$

we get similar results. If we let α, β, γ, and δ be the roots of $x^4 - x^3 - x^2 - x - 1 = 0$, then

$$Q_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + \gamma\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + \cdots + \gamma^n + \delta Q_{n-1}. \tag{3.7}$$

In multisecting the Quadranacci convolution array,

$$G_k(x) = \frac{(1 - \alpha^k x^k)(1 - \beta^k x^k)(1 - \gamma^k x^k)(1 - \delta^k x^k)}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)},$$

where $G_k(x)$ is the numerator polynomial from which the generating functions can be derived for multisecting the Quadranacci convolution sequences.

We can derive the following from (3.7):

$$6Q_n - 3Q_{n-1} - Q_{n-2} = \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\beta^n - \gamma^n}{\beta - \gamma} + \frac{\gamma^n - \delta^n}{\gamma - \delta} + \frac{\delta^n - \alpha^n}{\delta - \alpha} + \frac{\beta^n - \delta^n}{\beta - \delta} + \frac{\alpha^n - \gamma^n}{\alpha - \gamma}. \tag{3.8}$$

4. GENERALIZED FIBONACCI AND LUCAS NUMBERS

Start with

$$f(x) = \prod_{i=1}^{m} (x - \alpha_i);$$

then if

$$f(x) = x^m - x^{m-1} - x^{m-2} - \cdots - 1,$$

in particular, then

$$\frac{1}{s!} \cdot \frac{f^{(s)}(x)}{f(x)} = \prod \frac{1}{(x - \alpha_{i_1})(x - \alpha_{i_2}) \ldots (x - \alpha_{i_s})}$$

$$1 \le i_1 < i_2 < i_3 < \cdots < i_s \le m$$

over all subscripts restrained above.

If $s = m$, then we get, after some effort,

$$\frac{x}{1 - x - x^2 - \cdots - x^m} = \sum_{n=0}^{\infty} F_n^* x^n, \tag{4.1}$$

where F_n^* are the generalized Fibonacci numbers of the preceding section.

If $s = m$, we get the corresponding Lucas numbers

$$\mathcal{L}_n = \alpha_1^n + \alpha_2^n + \cdots + \alpha_m^n.$$

But, for those $1 < s < m$ we get other generalized Fibonacci sequences with some interesting properties studies by Chow [3]. We note two quick theorems.

Theorem 4.1:

Let

$$f(x) = \prod_{i=1}^{m} (x - \alpha_i), \; m \ge 2.$$

Then $\{\mathcal{L}_n\} = \{m, 1, 3, 7, 15, 32, \ldots\}$ for m terms. That is,

$$\mathcal{L}_0 = m,\; \mathcal{L}_1 = 2^1 - 1,\; \mathcal{L}_2 = 2^2 - 1,\; \ldots,\; \mathcal{L}_s = 2^s - 1,\; \ldots,\; \mathcal{L}_m = 2^m - 1.$$

After m terms, the recurrence takes over. In fact, $\mathcal{L}_m$ is the first term yielded by the recurrence. Further,

Theorem 4.2: The generating function for $\{\mathcal{L}_n\}$ is

$$\frac{mx - (m-1)x^2 - (m-2)x^3 - \cdots - x^m}{1 - x - x^2 - \cdots - x^m} = \sum_{n=0}^{\infty} \mathcal{L}_n x^n. \tag{4.2}$$

Using the observation that

$$G_m(x) + x \approx G_{m+1}(x)$$

For $(m+1)$ terms, one can then get an inductive proof for the starting values theorem. Of course, one has a starting values theorem for the regular generalized Fibonacci numbers in generalized Pascal triangles, and these are $1, 1, 2, 2^2, 2^3, \ldots$, until we reach the full length of the recurrence. Of great interest, of course, are those of the form

$$\frac{kx - x^2}{1 - x - x^2 - \cdots - x^m},$$

which starts off $k, k-1, 2k-1, \ldots$, which now double until the recurrence takes over.

For $s = 2$,

$$\begin{aligned}\frac{1}{2!} \cdot \frac{f^{(2)}(x)}{f(x)} &= \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha^n - \gamma^n}{\alpha - \gamma} + \cdots + \frac{\beta^n - \gamma^n}{\beta - \gamma} + \cdots \\ &= \frac{T_{m-1}x - T_{m-2}x^2 - \cdots - x^{m-1}}{1 - x - x^2 - x^3 - \cdots - x^m},\end{aligned}$$

where the T_m are the triangular numbers.

If one attempts to multisect the generalized Fibonacci numbers, one needs, of course, the generalized Lucas numbers in the recurrence relation. Recapping our results so far, we list each auxiliary polynomial:

$$\begin{array}{lll} m = 2 & \text{Fibonacci} & L_k = \alpha^k + \beta^k \\ & & x^2 - L_k x + (-1)^k \\[6pt] m = 3 & \text{Tribonacci} & S_k = \alpha^k + \beta^k + \gamma^k \\ & & x^3 - S_k x^2 + S_{-k} x - 1 \\[6pt] m = 4 & \text{Quadranacci} & S_k = \alpha^k + \beta^k + \gamma^k + \delta^k \\ & & x^4 - S_k x^3 + \frac{1}{2}(S_k^2 - S_{2k})x^2 - S_{-k}x + 1 \end{array}$$

What is involved, then, are the elementary symmetric functions for the original polynomial but for the kth powers of the roots.

5. GENERALIZED LUCAS NUMBERS AND SYMMETRIC FUNCTIONS OF kTH POWERS

If

$$x^m + c_1 x^{m-1} + c_2 x^{m-2} + \cdots + c_m = 0$$

has roots $\alpha_1, \alpha_2, \ldots, \alpha_m$, and $S_k = \alpha_1^k + \alpha_2^k + \cdots + \alpha_m^k$, then

$$c_k = \frac{(-1)^k}{k!} \begin{vmatrix} S_1 & 1 & 0 & 0 & \cdots \\ S_2 & S_1 & 2 & 0 & \cdots \\ S_3 & S_2 & S_1 & 3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & k-1 \\ S_k & S_{k-1} & \cdots & \cdots & S_1 \end{vmatrix} \tag{5.1}$$

which stems from the system of equations

$$
\begin{aligned}
&S_1 + c_1 &&= 0\\
&S_2 + c_1S_1 + 2c_2 &&= 0\\
&S_3 + c_1S_2 + c_2S_1 + 3c_3 &&= 0\\
&S_4 + c_1S_3 + c_2S_2 + c_3S_1 + 4c_4 &&= 0\\
&\qquad \cdots &&\cdots
\end{aligned}
\tag{5.2}
$$

which are Newton's Identities as given by Conkwright [4].

If you look at these equations, you have four unknowns c_1, c_2, c_3, and c_4 if S_1, S_2, S_3, and S_4 are given. Thus, you can treat this as a nonhomogeneous system and hence solve for c_1, c_2, c_3, or c_4, but strangely enough, while working, this does not yield the clever expression first given.

Consider instead

$$
\begin{aligned}
c_0S_1 + c_1 &= 0\\
c_0S_2 + c_1S_1 + 2c_2 &= 0\\
c_0S_3 + c_1S_2 + c_2S_1 &= -3c_3
\end{aligned}
$$

where $c_0 = 1$. Solve the system for c_0 by Cramer's rule:

$$
c_0 = 1 = \frac{\begin{vmatrix} 0 & 1 & 0\\ 0 & S_1 & 2\\ -3c_3 & S_2 & S_1 \end{vmatrix}}{\begin{vmatrix} S_1 & 1 & 0\\ S_2 & S_1 & 2\\ S_3 & S_2 & S_1 \end{vmatrix}} = \frac{-3!c}{\begin{vmatrix} S_1 & 1 & 0\\ S_2 & S_1 & 2\\ S_3 & S_2 & S_1 \end{vmatrix}}
$$

or

$$
c_3 = \frac{(-1)^3}{3!}\begin{vmatrix} S_1 & 1 & 0\\ S_2 & S_1 & 2\\ S_3 & S_2 & S_1 \end{vmatrix}.
$$

From (5.1) one can sequentially find $c_1, c_2, \ldots, c_k$ given $S_1, S_2, \ldots, S_k$, but this soon becomes untractable in practice.

However, we can make a new representation of the generalized Lucas sequences by using the set of equations (5.2) to derive

$$
S_k = (-1)^k \begin{vmatrix}
1c_1 & 1 & 0 & 0 & 0 & \cdots & 0\\
2c_2 & c_1 & 1 & 0 & 0 & \cdots & 0\\
3c_3 & c_2 & c_1 & 1 & 0 & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
kc_k & c_{k-1} & c_{k-2} & \cdots & \cdots & \cdots & c_1
\end{vmatrix}.
\tag{5.3}
$$

We rewrite (5.2) as

$$
\begin{aligned}
&(1)c_1 + S_1 &&= 0\\
&(1)2c_2 + S_1c_1 + S_2 &&= 0\\
&(1)3c_3 + S_1c_2 + S_2c_1 + S_3 &&= 0\\
&(1)4c_4 + S_1c_3 + S_2c_2 + S_3c_1 + S_4 &&= 0
\end{aligned}
$$

Here, again, we have a known variable (1) which we solve for using Cramer's rule for the nonhomogeneous set of equations, as

$$1 = \frac{\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & c_1 & 1 & 0 \\ 0 & c_2 & c_1 & 1 \\ -S_4 & c_3 & c_2 & c_1 \end{vmatrix}}{\begin{vmatrix} 1c_1 & 1 & 0 & 0 \\ 2c_2 & c_1 & 1 & 0 \\ 3c_3 & c_2 & c_1 & 1 \\ 4c_4 & c_3 & c_2 & c_1 \end{vmatrix}} \quad \frac{S_4}{\begin{vmatrix} 1c_1 & 0 & 0 & 0 \\ 2c_2 & c_1 & 1 & 0 \\ 3c_3 & c_2 & c_1 & 1 \\ 4c_4 & c_3 & c_2 & c_1 \end{vmatrix}}$$

or

$$S_4 = (-1)^4 \begin{vmatrix} c_1 & 1 & 0 & 0 \\ 2c_2 & c_1 & 1 & 0 \\ 3c_3 & c_2 & c_1 & 1 \\ 4c_4 & c_3 & c_2 & c_1 \end{vmatrix}.$$

Considering where these problems came from, if $c_1 = c_2 = -1$, $c_k = 0$ for $k > 2$, then $S_k = L_k$, the familiar Lucas numbers, which are then given by a tri-diagonal continuant,

$$L_k = (-1)^k \begin{vmatrix} -1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -2 & -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{vmatrix},$$

while the generalized Lucas sequence related to the Tribonacci numbers is given by the quadradiagonal continuant,

$$\mathscr{L}_k = (-1)^k \begin{vmatrix} -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ -2 & -1 & 1 & 0 & 0 & \cdots & 0 \\ -3 & -1 & -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & -1 & -1 & 1 & \cdots & 0 \\ 0 & 0 & -1 & -1 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{vmatrix}.$$

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SOME RESTRICTED MULTIPLE SUMS

LEONARD CARLITZ
Duke University, Durham, NC 27706

1. INTRODUCTION

Let a, b be positive integers, $(a,b) = 1$. Consider the sum

$$S = \sum_{br+as<ab} x^{br+as} \equiv \sum_{\substack{r=0 \\ br+as<ab}}^{a-1} \sum_{s=0}^{b-1} x^{br+as}. \tag{1.1}$$

We will show that

$$S = \frac{1 - x^{ab}}{(1 - x^a)(1 - x^b)} - \frac{x^{ab}}{1 - x}. \tag{1.2}$$

As an application of (1.2), let $B_n(x)$ denote the Bernoulli polynomial of degree n defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!},\quad B_n = B_n(0).$$

Then we have

$$\sum_{br+as<ab} B_n\left(x + \frac{r}{a} + \frac{s}{b}\right) = (B(ab) + abB(x))^n - (aB + bB(ax))^n, \tag{1.3}$$

where

$$(uB(x) + vB(y))^n \equiv \sum_{k=0}^{n} \binom{n}{k} u^k v^{n-k} B_k(x) B_{n-k}(y).$$

We also evaluate the sum

$$\sum_{br+as<ab} (x + br + as)^n \tag{1.4}$$

in terms of Bernoulli polynomials; see (3.8) below.

Let a, b, c be positive integers such that $(b,c) = (c,a) = (a,b) = 1$. The sum (1.1) suggests the consideration of the sums

$$S_1 = \sum_{bcr+cas+abt<abc} x^{bcr+cas+abt}$$

and

$$S_2 = \sum_{bcr+cas+abt<2abc} x^{bcr+cas+abt}$$

where $0 \leq r < a$, $0 \leq s < b$, $0 \leq t < c$. We are unable to evaluate S_1 and S_2 separately. However, we show that