SUMMATION OF THE SERIES $y^n + (y+1)^n + \cdots + x^n$

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In 1970, Levy [1] published a number of results concerning the sum of the series $1^n + 2^n + \cdots + x^n$, which is known to be an $n+1$-degree polynomial $P_n(x)$ whenever x is a positive integer. However, there is a natural generalization that will also hold for negative integers and zero as well. This is given in the following theorem.

Theorem 1: For each positive integer n there is exactly one polynomial such that

$$\sum_{k=y+1}^{x} k^n = P_n(x) - P_n(y)$$

for all integral values of x and y, where $y < x$.

This theorem also holds for $n = 0$ if 0^0 is interpreted as 1. The proof follows easily from two lemmas.

Lemma 1: For each integer value of $x \geq 0$,

$$\sum_{k=1}^{x} k^n = P_n(x) - P_n(0).$$

This is true because $P_n(0) = 0$ for all n.

Lemma 2: For each integer value of $y < 0$,

$$\sum_{k=y+1}^{0} k^n = P_n(0) - P_n(y).$$

Proof:

$$\sum_{k=y+1}^{0} k^n = \sum_{j=0}^{-y-1} (-j)^n = (-1)^n P_n(-y-1) = -P_n(y),$$

where the last equality follows from Theorem 3 in the paper by Levy. When x is a positive integer, $P_n(x)$ is the sum of the series from 1 to n, and when x is a negative integer, then $-P_n(x)$ is the sum of the series from $x+1$ to 0.

REFERENCES

1. L. S. Levy. "Summation of the Series $1^n + 2^n + \cdots + x^n$ Using Elementary Calculus." *American Math. Monthly* 77, No. 8 (Oct. 1970):840-852.

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