ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E., Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within 4 months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$; $L_0 = 2$, $L_1 = 1$. Also a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-430 *Proposed by M. Wachtel, H. Klauser, and E. Schmutz, Zürich, Switz.*

For every positive integer a, prove that

$$(a^2 + a - 1)(a^2 + 3a + 1) + 1$$

is a product $m(m + 1)$ of two consecutive integers.

B-431 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA.*

For which fixed ordered pairs (h, k) of integers does

$$F_n(L_{n+h}^2 - F_{n+h}^2) = F_{n+4}(L_{n+k}^2 - F_{n+k}^2)$$

for all integers n?

B-432 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA.*

Let $G_n = F_n F_{n+3}^2 - F_{n+2}^3$. Prove that the terms of the sequence

$$G_0,\ G_1,\ G_2,\ \ldots$$

alternate in sign.

B-433 *Proposed by J. F. Peters and R. Pletcher, St. John's University, Collegeville, MN.*

For each positive integer n, let q_n and r_n be the integers with

$$n = 3q_n + r_n \quad \text{and} \quad 0 \le r_n < 3.$$

Let $\{T(n)\}$ be defined by

$$T(0) = 1,\ T(1) = 3,\ T(2) = 4, \text{ and } T(n) = 4q_n + T(r_n) \text{ for } n \ge 3.$$

Show that there exist integers a, b, c such that

$$T(n) = \left[\frac{an + b}{c}\right],$$

where $[x]$ denotes the greatest integer in x.

B-434 *Proposed by Herta T. Freitag, Roanoke, VA.*

For which positive integers n, if any, is

$$L_{3n} - (-1)^n L_n$$

a perfect square?

B-435 *Proposed by M. Wachtel, H. Klauser, and E. Schmutz, Zürich, Switz.*

For every positive integer a, prove that no integral divisor of

$$a^2 + a - 1$$

is congruent to 3 or 7, modulo 10.

SOLUTIONS

First Term as GCD

B-406 *Proposed by Wray G. Brady, Slippery Rock State College, PA.*

Let $x_n = 4L_{3n} - L_n^3$ and find the greatest common divisor of the terms of the sequence $x_1, x_2, x_3, \ldots$.

Solution by Paul S. Bruckman, Concord, CA.

$$\begin{aligned} x_n &= L_n(4L_{3n}/L_n - L_n^2) = L_n[4a^{2n} - 4(ab)^n + 4b^{2n} - a^{2n} - 2(ab)^n - b^{2n}] \\ &= 3L_n[a^{2n} - 2(ab)^n + b^{2n}] = 3L_n(a^n - b^n)^2 = 15L_nF_n^2, \end{aligned}$$

or

$$x_n = 15F_nF_{2n},\ n = 1, 2, 3, \ldots .$$

Note that $x_1 = 15F_1F_2 = 15$. Hence, $x_1 | x_n$, $n = 1, 2, 3, \ldots$. It follows that the greatest common divisor of $\{x_n\}$ is $x_1 = 15$.

Also solved by Herta T. Freitag, John W. Milsom, Bob Prielipp, E. Schmutz, A. G. Shannon, Sahib Singh, Lawrence Somer, M. Wachtel, Gregory Wulczyn, and the proposer.

Generator of Pascal Triangle

B-407 *Proposed by Robert M. Giuli, University of California, Santa Cruz, CA.*

Given that

$$\frac{1}{1 - x - xy} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{nk} x^n y^k$$

is a double ordinary generating function for a_{nk}; determine a_{nk}.

Solution by Paul S. Bruckman, Concord, CA.

$$\begin{aligned} (1 - x - xy)^{-1} &= (1 - x(1 + y))^{-1} = \sum_{n=0}^{\infty} x^n (1 + y)^n = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{n}{k} y^k \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} x^n y^k. \end{aligned}$$

The binomial coefficient $\binom{n}{k}$ is defined to be zero for $k > n$. Hence, we may extend the second sum above over all nonnegative k, i.e.,

$$(1 - x - xy)^{-1} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n}{k} x^n y^k.$$

Thus,

$$a_{nk} = \binom{n}{k}, \ (n,\ k = 0,\ 1,\ 2,\ \ldots).$$

Also solved by W. O. J. Moser, A. G. Shannon, Sahib Singh, and the proposer.

Proposal Tabled

B-408 *No solutions received.*

Exact Divisor

B-409 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Let $P_n = F_n F_{n+a}$. Must $P_{n+6r} - P_n$ be an integral multiple of

$$P_{n+4r} - P_{n+2r}$$

for all nonnegative integers a and r?

Solution by Sahib Singh, Clarion State College, Clarion, PA.

Yes. Using

$$L_n = a^n + b^n,\ F_n = \frac{a^n - b^n}{a - b},$$

we see that divisibility of

$$P_{n+6r} - P_n \text{ by } P_{n+4r} - P_{n+2r}$$

is equivalent to divisibility of

$$L_{2n+12r+a} - L_{2n+a} \text{ by } L_{2n+8r+a} - L_{2n+4r+a}.$$

The result follows immediatly by seeing that

$$L_{2n+12r+a} - L_{2n+a} = (L_{2n+8r+a} - L_{2n+4r+a})(L_{4r} + 1).$$

Also solved by Paul S. Bruckman, Bob Prielipp, and the proposer.

Golden Limit

B-410 *Proposed by M. Wachtel, Zürich, Switz.*

Some of the solutions of

$$5(x^2 + x) + 2 = y^2 + y$$

in positive integers x and y are:

$$(x,\ y) = (0,\ 1),\ (1,\ 3),\ (10,\ 23),\ (27,\ 61).$$

Find a recurrence formula for the x_n and y_n of a sequence of solutions $(x_n,\ y_n)$. Also find $\lim(x_{n+1}/x_n)$ and $\lim(x_{n+2}/x_n)$ as $n \to \infty$ in terms of $\alpha = (1 + \sqrt{5})/2$.

Solution by Paul S. Bruckman, Concord, CA.

Multiplying the given Diophantine equation throughout by 4, completing the square, and simplifying yields:

(1) $$Y^2 - 5X^2 = 4,$$

where

(2) $$X = 2x + 1,\ Y = 2y + 1.$$

The solutions of (1) in positive integers are known to be

(3) $$(X_m,\ Y_m) = (F_{2m},\ L_{2m})_{m=0}^{\infty}.$$

However, due to (2), X and Y must also be odd. By inspection of the first few values (mod 3) of the Fibonacci and Lucas sequences, it is apparent that these values are even if and only if their subscripts are multiples of 3. Hence, we must have $m \equiv \pm 1 \pmod 3$ in (3). Distinguishing between these cases, we obtain two distinct sets of solutions:

(4) $$(X_m^{(1)},\ Y_m^{(1)}) = (F_{6m+2},\ L_{6m+2})_{m=0}^{\infty};$$

(5) $$(X_m^{(2)},\ Y_m^{(2)}) = (F_{6m+4},\ L_{6m+4})_{m=0}^{\infty}.$$

In terms of the original problem, this yields the following distinct solution sequences:

(6) $$(x_n^{(1)},\ y_n^{(1)}) = \left\{\tfrac{1}{2}(F_{6n+2} - 1),\ \tfrac{1}{2}(L_{6n+2} - 1)\right\}_{n=0}^{\infty};$$

(7) $$(x_n^{(2)},\ y_n^{(2)}) = \left\{\tfrac{1}{2}(F_{6n+4} - 1),\ \tfrac{1}{2}(L_{6n+4} - 1)\right\}_{n=0}^{\infty}.$$

It is apparent from the fact that the successive indices of the Fibonacci and Lucas sequences in (6) and (7) "increase by sixes," that we are interested in the second-order equation for a^6, which must be the same for b^6. Since $a^6 = 8a + 5$ and $a^{12} = 144a + 89$ (special cases of $a^r = aF_r + F_{r-1}$), it is evident that a and b satisfy the common equation:

(8) $$z^{12} - 18z^6 + 1 = 0.$$

Let

(9) $$D_n = z_{n+2} - 18z_{n+1} + z_n, \text{ where } z_n = x_n^{(k)} \text{ or } y_n^{(k)},\ k = 1 \text{ or } 2.$$

We see from (6) and (7) that $D_n = \frac{1}{2}(-1 + 18 - 1)$ [using (8)], or

(10) $$D_n = 8,\ n = 0,\ 1,\ 2,\ \ldots\ .$$

This is a recursion for the $x_n^{(k)}$ and $y_n^{(k)}$, as required.

A *homogeneous* recursion may be obtained by noting simply that

$$D_{n+1} - D_n = 0.$$

This is equivalent to the following third-order recursion:

(11) $$z_{n+3} - 19z_{n+2} + 19z_{n+1} - z_n = 0,\ n = 0,\ 1,\ 2,\ \ldots\ .$$

It is evident from (6) and (7) that

(12) $$\lim_{n\to\infty} x_{n+1}^{(k)}/x_n^{(k)} = \lim_{n\to\infty} y_{n+1}^{(k)}/y_n^{(k)} = a^6,$$

and

(13) $$\lim_{n\to\infty} x_{n+2}^{(k)}/x_n^{(k)} = \lim_{n\to\infty} y_{n+2}^{(k)}/y_n^{(k)} = a^{12}\ (k = 1 \text{ or } 2).$$

Also solved by the proposer.

Tridiagonal Determinants

B-411 *Proposed by Bart Rice, Crofton, MD.*

Tridiagonal n by n matrices $A_n = (a_{ij})$ of the form

$$a_{ij} = \begin{cases} 2a\ (a \text{ real}) & \text{for } j = i \\ 1 & \text{for } j = i \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

occur in numerical analysis. Let $d_n = \det A_n$.

(i) Show that $\{d_n\}$ satisfies a second-order homogeneous linear recursion.

(ii) Find closed-form and asymptotic expressions for d_n.
(iii) Derive the combinatorial idantity

$$\sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1}(-x)^k = (x+1)^{(n-1)/2}\frac{\sin rn}{\sin r}, \quad x > 0, \quad r = \tan^{-1}\sqrt{x}.$$

Solution by Paul S. Bruckman, Concord, CA.

We see that

$$A_n = \begin{pmatrix} 2a & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2a & 1 & 0 & 0 & \cdots \\ 0 & 1 & 2a & 1 & 0 & \cdots \\ 0 & 0 & 1 & 2a & 1 & \cdots \\ 0 & 0 & 0 & 1 & 2a & \cdots \\ 0 & 0 & 0 & 0 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}_{n \times n} \tag{1}$$

Taking determinants along the first row, we find that $d_n = 2ad_{n-1} - \det B_n$, where

$$B_n = \begin{pmatrix} 1 & 1 \quad 0 \quad 0 \quad \cdots \quad 0 \\ 0 & \\ 0 & \\ \vdots & \Big(\qquad A_{n-2} \qquad \Big) \\ \vdots & \\ 0 & \end{pmatrix}_{(n-1)\times(n-1)}$$

Taking the determinant of B_n along its first column, we see easily that det $B_n = d_{n-2}$. Hence, we have the following recursion:

$$d_{n+2} - 2ad_{n+1} + d_n = 0, \quad n = 1, 2, 3, \ldots . \tag{3}$$

Note also the initial values of the recursion:

$$d_1 = 2a, \quad d_2 = 4a^2 - 1. \tag{4}$$

The characteristic polynomial of (3) is

$$p(x) = x^2 - 2ax + 1, \tag{b}$$

which has the zeros

$$u = a + \sqrt{a^2 - 1}, \quad v = a - \sqrt{a^2 - 1}. \tag{6}$$

It follows that d_n is of the form $pu^n + qv^n$, for some constants p and q which are determined from the initial conditions. Note that

$$uv = 1, \quad u + v = 2a.$$

We then find

$$d = \frac{u^{n+1} - v^{n+1}}{u - v}, \quad n = 1, 2, 3, \ldots . \tag{7}$$

The behavior of d_n as $n \to \infty$ depends on the magnitude of a, and we distinguish several cases.

Case I: $0 \le |a| < 1$.

Let $a = \cos\theta$. Then $u = e^{i\theta}$, $v = e^{-i\theta}$, and so

$$d_n = \sin(n+1)\theta/\sin\theta. \tag{8}$$

In this case, the sequence (d_n) is dense in

$$(-\csc\theta,\ \csc\theta) \equiv (-(1-a^2)^{-\frac{1}{2}},\ (1-a^2)^{-\frac{1}{2}})$$

and oscillates within this interval without lending itself to approximation by an asymptotic expression.

Case II: $a = -1$.

Then $u = v = -1$. Since

$$d_n = \sum_{k=0}^{n} u^{n-k}v^k,$$

thus

$$d_n = \sum_{k=0}^{n}(-1)^k = \tfrac{1}{2}(1+(-1)^n) = \begin{cases}1, & n \text{ even}\\ 0, & n \text{ odd}\end{cases}.$$

Clearly, d_n oscillates between these two values only, in this case.

Case III: $a = 1$.

Here $u = v = 1$. Hence,

$$d_n = \sum_{k=0}^{n} u^{n-k}v^k = \sum_{k=0}^{n} 1 = n+1.$$

Therefore, for this case, $d_n \sim n$ as $n \to \infty$.

Case IV: $a < -1$.

Then $v < -1 < u < 0$, which implies $u^n \to 0$ as $n \to \infty$. Hence

$$d_n \sim \frac{v^{n+1}}{v-u} \text{ as } n \to \infty.$$

Case V: $a > 1$.

Then $0 < v < 1 < u$, which implies $v^n \to 0$ and $d_n \sim \dfrac{u^{n+1}}{u-v}$ as $n \to \infty$.

To prove the given combinatorial identity, note that

$$\tfrac{1}{2}(u-v)d_{n-1} = \tfrac{1}{2}(u^n - v^n) = \tfrac{1}{2}\sum_{k=0}^{n}\binom{n}{k}a^{n-k}(a^2-1)^{\frac{1}{2}k}(1-(-1)^k)$$

$$= \sum_{k=0}^{[\frac{1}{2}(n-1)]}\binom{n}{2k+1}a^{n-2k-1}(a^2-1)^{k+\frac{1}{2}}$$

$$= \tfrac{1}{2}(u-v)a^{n-1}\sum_{k=0}^{[\frac{1}{2}(n-1)]}\binom{n}{2k+1}\left(\frac{1-a^2}{a^2}\right)^k(-1)^k, \text{ or}$$

$$d_{n-1} = a^{n-1}\sum_{k=0}^{[\frac{1}{2}(n-1)]}\binom{n}{2k+1}\left(\frac{1-a^2}{a^2}\right)^k(-1)^k. \tag{9}$$

In (9), let $x = a^{-2} - 1$, supposing Case I above, so that we can have $x > 0$. Then, since $r = \tan^{-1}\sqrt{x}$ is in $(0, \frac{1}{2}\pi)$,

$$r = \tan^{-1}\{(1-a^2)^{\frac{1}{2}}/|a|\} = \cos^{-1}\{|a|\}.$$

Also, $\sin r = (1-a^2)^{\frac{1}{2}}$. Using the notation of Case I, $r = \theta$ if $a > 0$ and

$r = \pi - \theta$ if $a < 0$. In either case, $\sin\theta = \sin r$. Thus, using (8),

$$d_{n-1} = \sin n\theta/\sin\theta = \begin{cases} \sin nr/\sin r, & \text{if } a > 0; \\ (-1)^{n-1}\sin nr/\sin r, & \text{if } a < 0. \end{cases}$$

Also, $a^2 = (1 + x)^{-1}$, which implies that

$$a^{n-1} = \begin{cases} (1 + x)^{-\frac{1}{2}(n-1)}, & a > 0; \\ (-1)^{n-1}(1 + x)^{-\frac{1}{2}(n-1)}, & a < 0. \end{cases}$$

In either case, it follows from (9) that

$$(1 + x)^{-\frac{1}{2}(n-1)} \sum_{k=0}^{[\frac{1}{2}(n-1)]} \binom{n}{2k + 1}(-x)^k = \sin rn/\sin r,$$

which is equivalent to the desired identity.

Also solved by the proposer.

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