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ON THE CONVERGENCE OF ITERATED EXPONENTIATION—I

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We have investigated the properties of the function $f(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ with an infinite number of x's in the region $0 < x < e^{1/e}$. We have also defined a class of functions $F_n(x)$ which are a generalization of $f(x)$, and which exhibit the property of "dual convergence," i.e., convergence to different values of $F_n(x)$ as $n \to \infty$, depending upon whether n is even or odd.

An elementary exercise is to find a positive x satisfying

(1) $$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2$$

when an infinite number of exponentiations is understood [1], [2]. The standard solution is to note that the exponent of the first x must be 2, and thus $x = \sqrt{2}$. Indeed, the sequence f_n defined by

(2) $$f_0 = 1$$
$$f_{n+1} = 2^{f_n/2}$$

does converge to 2 as n goes to infinity. Now consider the problem

(3) $$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = \frac{1}{3}.$$

By analogy, one might assume that

$$x = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

is the solution; however, this is too naive because the sequence f_n defined by

(4) $$f_0 = 1$$
$$f_{n+1} = \left(\frac{1}{27}\right)^{f_n}$$

does not converge.

The purpose of this article is to discuss some criteria for convergence of sequences of the form

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(5) $$f_n = g_1^{g_2^{\cdot^{\cdot^{\cdot^{g_n}}}}}$$

where g_i is some given sequence of positive numbers. Applying these criteria to the case where $g_i = x$ for all i, we will show convergence of the resulting sequence for x in the range

$$\left[\left(\frac{1}{e}\right)^e,\ e^{\left(\frac{1}{e}\right)}\right]$$

where e is the base of the natural logarithm. For x larger than $e^{\frac{1}{e}}$, the sequence f_n diverges to infinity, while for x in the range $(0,\ e^{-e})$ the even and odd sequences f_{2n} and f_{2n+1} both converge, but to different values. This property of "dual convergence" occurs for many starting sequences g_j, some of which we will discuss briefly.

Before proceeding, we should comment on the order in which the exponentiations of equation (5) are to be carried out. Rather than insert cumbersome parentheses, we will understand throughout this paper that this expression is to be evaluated "from the top down." More precisely g_{n-1} is taken to the g_nth power, g_{n-2} is taken to the resulting power, and so on. The only other simple specification of the ordering of the exponentiations is "from the bottom up," but this merely reduces to g_1 raised to the product of the remaining g's.

It is convenient at this point to introduce a shorthand notation for expression of the form in equation (5). We thus write for $m \geq n$,

(6) $$\mathop{\Xi}_{j=n}^{m} g_j = g_n^{g_{n+1}^{\cdot^{\cdot^{\cdot^{g_m}}}}}$$

A simple recursive definition of this quantity is

(7) $$\mathop{\Xi}_{j=n}^{m} g_j = \begin{cases} g_n, & n = m \\ \exp\left\{\left(\mathop{\Xi}\limits_{j=n+1}^{m} g_j\right)\cdot \log g_n\right\}, & m > n \end{cases}$$

We now prove two theorems on the convergence of these sequences.

Theorem 1: If there exists a positive integer i such that for all $j \geq i$ we have $1 \leq g_j \leq e^{\frac{1}{e}}$, then the sequence $\mathop{\Xi}\limits_{j=1}^{n} g_j$ converges as $n \to \infty$.

Proof: When $n > i$, we have

(8) $$\mathop{\Xi}_{j=1}^{n} g_j = g_1^{\cdot^{\cdot^{\cdot^{g_{i-1}^{\left(\mathop{\Xi}\limits_{j=i}^{n} g_j\right)}}}}}$$

consequently, we need only prove the theorem when all g_i lie in the range $\left[1,\ e^{\frac{1}{e}}\right]$. In this case, $\mathop{\Xi}\limits_{j=1}^{n} g_j$ is easily shown to be a monotonic increasing function of any g_i. This, in turn, implies $\mathop{\Xi}\limits_{j=1}^{n} g_j > \mathop{\Xi}\limits_{j=1}^{n-1} g_j$; i.e., we have an increasing sequence. However, the sequence is also bounded because

(9) $$\mathop{\Xi}_{j=1}^{n} g_j \le \mathop{\Xi}_{1}^{n}\left(e^{\frac{1}{e}}\right) < \left(e^{\frac{1}{e}}\right)^{\cdot^{\cdot^{\cdot^{\left(e^{\frac{1}{e}}\right)^{e}}}}} = e.$$

Now, by an elementary theorem [3], any bounded and monotonic sequence is convergent.

Theorem 2: If there exists a positive integer i such that for all $j > i$ we have $0 < g_j \le 1$, then the even and odd sequences

$$\mathop{\Xi}_{j=1}^{2n} g_j \quad \text{and} \quad \mathop{\Xi}_{j=1}^{2n+1} g_j$$

are both convergent as $n \to \infty$.

Proof: Again, we need only prove the theorem when all g_i are in the range $[0, 1]$. Also, we need only consider the even sequence because the odd sequence is merely g_1 raised to an even sequence. Now for x and y in the range $[0, 1]$ the quantity x^y is a monotonic decreasing function of y. Using this inductively on f_{2n}, we find f_{2n} is a monotonic decreasing function of g_{2n}. If we now replace g_{2n} with

$$g_{2n}^{\,g_{2n+1}^{\,g_{2n+2}}} > g_2$$

we can conclude that

(10) $$f_{2n+2} < f_{2n}.$$

However, f_{2n} is always bounded below by zero. Thus, we again have a monotonic bounded sequence which must converge.

With the help of these theorems we now return to the case $g_i = x$ independent of i. We state the result as a theorem.

Theorem 3: For positive x,

$$\mathop{\Xi}_{1}^{n} x \text{ converges as } n \to \infty \text{ iff } x \text{ lies in the interval } \left[\left(\frac{1}{e}\right)^{e},\ e^{\frac{1}{e}}\right].$$

Proof: For x in the interval $\left[1,\ e^{\frac{1}{e}}\right]$, Theorem 1 immediately implies convergence. For x larger than $e^{\frac{1}{e}}$ the sequence cannot converge because, if it did, it would converge to a solution f of the equation (see [4])

(11) $$x^f - f = 0.$$

Whenever $x > e^{\frac{1}{e}}$, the lefthand side of this equation is strictly positive for all real f and the equation has no solution. The curves of x versus f as obtained (see [2]) from equation (11) for $f < e$ are shown in Figures 1 and 2, which pertain to $x > 1$ and $x < 1$, respectively.

When $x < 1$ Theorem 2 applies and we have convergence of the even and odd sequences. Both these sequences must converge to solutions f of the equation

(12) $$x^{x^f} - f = 0.$$

We will now show that, for $\left(\frac{1}{e}\right)^{e} \le x < 1$, this equation has only one solution and therefore the even and odd sequences converge to the same number. Take the derivative of the lefthand side of equation (12) with respect to f,

(13) $$\frac{d}{df}(x^{x^f} - f) = \log^2 x \cdot x^f \cdot x^{x^f} - 1.$$

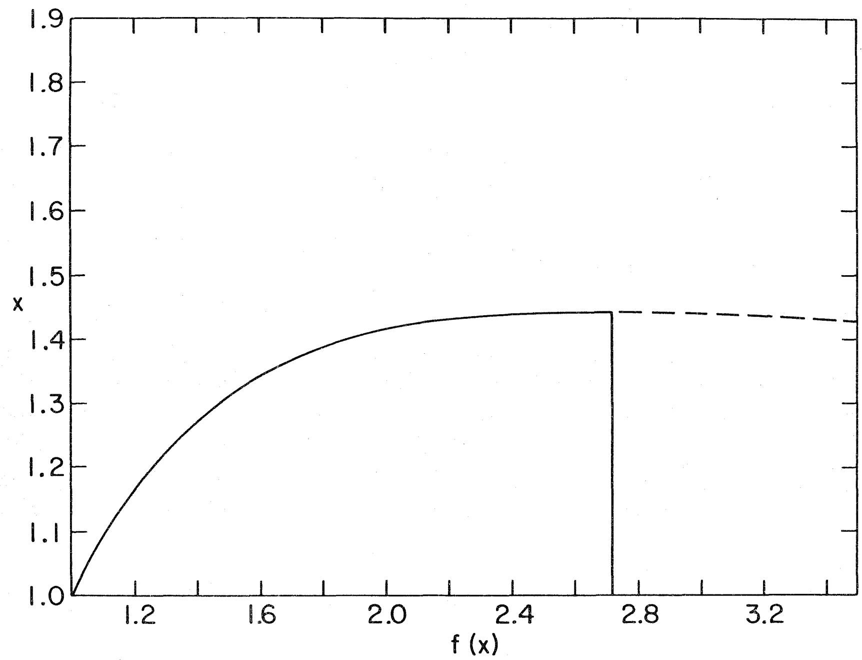

Fig. 1. The variable x as a function of $f(x)$, with $f(x)$ defined by (11), for values of $f(x)$ in the region $1 < f(x) < e$. The dashed part of the curve to the right of $f(x) = e$ is not meaningful.

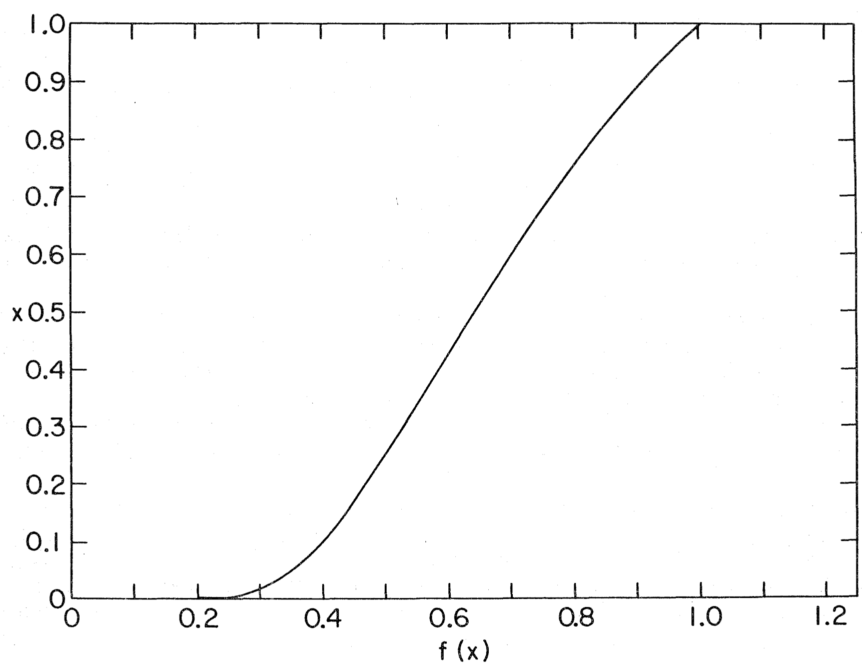

Fig. 2. The variable x as a function of $f(x)$, with $f(x)$ defined by (11) for values of $f(x)$ in the region $0 < f(x) < 1$.

Keeping $x < 1$ and maximizing the righthand side of (13) over f we obtain

(14) $$\frac{d}{df}(x^{x^f} - f) \leq -\frac{1}{e} \log x - 1.$$

If the righthand side of this inequality is negative, i.e., when

(15) $$1 > x \geq \left(\frac{1}{e}\right)^e,$$

then the quantity $x^{x^f} - f$ is a monotonic decreasing function of f and can only vanish at one point. This value of f is the number to which both the even and odd sequences must converge.

Finally we show by contradiction that $\mathop{\Xi}\limits_{1}^{n} x$ cannot converge for $x < \left(\frac{1}{e}\right)^e$. Assume it does converge to some number f which must satisfy (11). Define the sequence ε_n by

(16) $$\varepsilon_n = f_n - f.$$

In the proof of Theorem 2 we showed the even and odd sequences are both monotonic, and thus ε_n cannot vanish for finite n. The relation between ε_{n+1} and ε_n is

(17) $$\varepsilon_{n+1} = x^{f+\varepsilon_n} - f.$$

Expanding in powers of ε_n and using equation (11) gives

(18) $$\varepsilon_{n+1} = \varepsilon_n \log f + \mathcal{O}(\varepsilon_n^2).$$

Consequently the sequence cannot converge if $|\log f| > 1$ which corresponds to $x < \left(\frac{1}{e}\right)^e$. This completes the proof of Theorem 3.

We now return to the case of general g_j in equation (5). The above discussion of $g_j = x$ shows that under the conditions of Theorem 2, the limits of the even and odd sequences are not in general equal. The special role played by $\left(\frac{1}{e}\right)^e$ impels us to conjecture that the simple convergence of Theorem 1 may be extended for g_j in the range $\left[\left(\frac{1}{e}\right)^e, e^{\frac{1}{e}}\right]$, but we have no proof of this. Note that neither Theorem 1 nor 2 needs any assumption of the existence of a limit for g_j; this suggests it might be amusing to study g_j alternately inside and outside the above region.

In an informal report [5], we have studied several sequences where g_j goes to zero as j goes to infinity. In general upon iterated exponentiation these give rise to dual convergent sequences in the sense of Theorem 2, the even and odd sequences both converging to different numbers. As a particular example take $g_j = \frac{x}{j^2}$, and consider $\mathop{\Xi}\limits_{j=1}^{n} g_j$ as a function of x. In Figure 3, we have plotted this function versus x for $n = 10$ and 11. Increasing n further makes no visually discernible difference between the curves; even n essentially reproduce the $n = 10$ curve and odd n the $n = 11$ curve. Note the crossing points at $x = 1$ and 4 where one of the g_j is one and therefore the sequence converges after a finite number of steps.

In [5] we have also considered the sequence resulting from $g_j = jx$. Here g_j goes to infinity as j does; nonetheless, the resulting $\mathop{\Xi}\limits_{j=1}^{n} g_j$ converges as long as x is less than one. The amusing function resulting is piecewise continuous with discontinuities at $x = \frac{1}{k}$ where k is any positive integer. Three different values for $\mathop{\Xi}\limits_{j=1}^{n}(xj)$ are obtained by taking $x = \frac{1}{k}$ and $x = \frac{1}{k} \pm \varepsilon$ in the limit of vanishing ε.

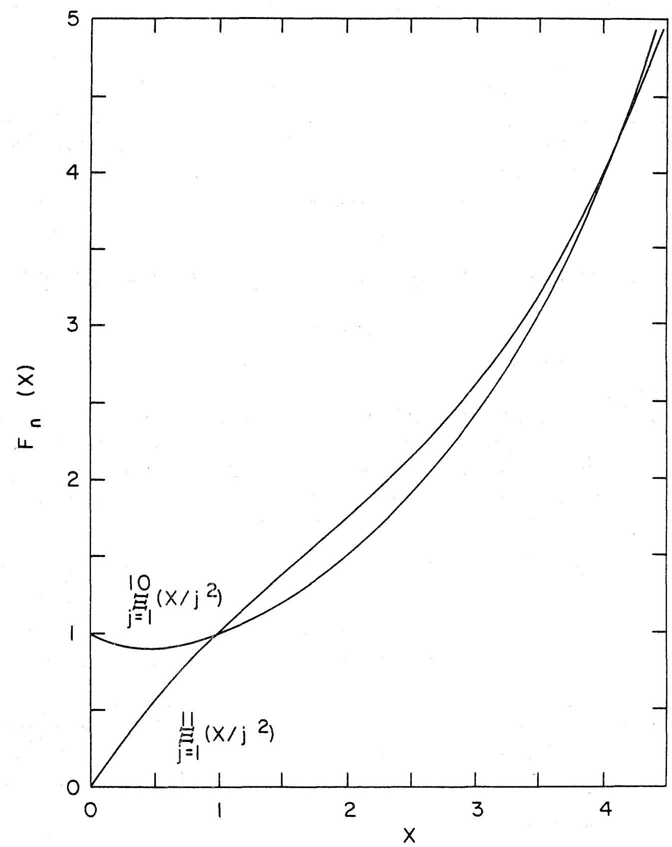

Fig. 3. The function $F_n(x) = \mathop{\Xi}\limits_{j=1}^{n}(x/j^2)$ *for* $n = 10$ *and* $n = 11$*, showing the dual convergence of* $F_n(x)$*. We note the "crossing points" at* $x = 1$ *and* $x = 4$*, where the two functions are equal.*

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Note: After completing this paper, we became aware of a similar calculation by Perry B. Wilson, in which some of the present results have been obtained (Stanford Linear Accelarator Report PEP-232, February 1977). We wish to thank Dr. S. Krinsky for calling our attention to this report.

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THE NUMBER OF PERMUTATIONS WITH A GIVEN NUMBER OF SEQUENCES

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1. Let $P(n, s)$ denote the number of permutations of $Z_n = \{1, 2, \ldots, n\}$ with s ascending or descending sequences. For example, the permutation 24315 has the ascending sequences 24, 15 and the descending sequence 431; the permutation 613254 has ascending sequences 13, 25 and descending sequences 61, 32, 54. André proved that $P(n, s)$ satisfies the recurrence

$$P(n+1, s) = sP(n, s) + 2P(n, s-1) + (n-s+1)P(n, s-2), \quad (n \geq 1), \tag{1.1}$$

where $P(0, s) = P(1, s) = \delta_{0,s}$; for proof see Netto [3, pp. 105-112].

Using (1.1), the writer [1] obtained the generating function

$$\sum_{n=0}^{\infty}(1 - x^2)^{-n/2}\frac{z^n}{n!}\sum_{s=0}^{\infty}P(n+1, s)x^{n-s} = \frac{1-x}{1+x}\left(\frac{\sqrt{1-x^2} + \sin z}{x - \cos z}\right)^2. \tag{1.2}$$

However, an explicit formula for $P(n, s)$ was not found.

In the present note, we obtain an explicit result, namely

$$\begin{cases} P(2n-1, 2n-s-2) = \displaystyle\sum_{j=1}^{n}(-1)^{n-j}2^{-j+2}(2j-1)!\overline{K}_{n,j}M_{n,j,s} \\ P(2n, 2n-s-1) = \displaystyle\sum_{j=1}^{n}(-1)^{n-j}2^{-j+1}(2j)!\overline{K}_{n,j}M_{n,j,s}, \end{cases} \tag{1.3}$$

where

$$\overline{K}_{n,j} = \frac{1}{(2j)!}\sum_{t=0}^{2j}(-1)^t\binom{2j}{t}(j-t)^{2n}$$

and

$$M_{n,j,s} = \sum_{t=0}^{n-j}(-1)^t\binom{n-j}{t}\binom{n-2}{s-t}.$$