ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E., Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1 \text{ and } L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-436 *Proposed by Sahib Singh, Clarion State College, Clarion, PA.*

Find an appropriate expression for the nth term of the following sequence and also find the sum of the first n terms:

$$4,\ 2,\ 10,\ 20,\ 58,\ 146,\ 388,\ 1010,\ \ldots\ .$$

B-437 *Proposed by G. Iommi Amunategui, Universidad Católica de Valparaíso, Valparaíso, Chile.*

Let $[m, n] = mn(m + n)/2$ for positive integers m and n. Show that:

(a) $[m + 1, n][m, n + 2][m + 2, n + 1] = [m, n + 1][m + 2, n][m + 1, n + 2]$.

(b) $\displaystyle\sum_{k=1}^{m} [m + 1 - k, k] = m(m + 1)^2(m + 2)/12$.

(We note that part (a) is the Hoggatt-Hansell "Star of David" property for the $[m, n]$.)

B-438 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Let n and w be integers with w odd. Prove or disprove the proposed identity

$$F_{n+2w}F_{n+w} - 2L_wF_{n+w}F_{n-w} + F_{n-w}F_{n-2w} = (L_{3w} - 2L_w)F_n^2.$$

B-439 *Proposed by A. P. Hillman, University of New Mexico, Albuquerque, NM.*

Can the proposed identity of B-438 be proved by mere verification for a finite set of ordered pairs (n, w)? If so, how few pairs suffice?

B-440 *Proposed by Jeffrey Shallit, University of California, Berkeley, CA.*

(a) Let $n = x^2 + y^2$, with x and y integers not both zero. Prove that there is a nonnegative integer k such that $n \equiv 2^k \pmod{2^{k+2}}$.

(b) If $n \equiv 2^k \pmod{2^{k+2}}$, must n be a sum of two squares?

B-441 *Proposed by Jeffrey Shallit, University of California, Berkeley, CA.*

A base-b palindrome is a positive integer whose base-b representation reads the same forward and backward. Prove that the sum of the reciprocals of all base-b palindromes converges for any given integer $b \geq 2$.

SOLUTIONS

GCD Not LCM

B-412 *Proposed by Phil Mana, Albuquerque, NM.*

Find the least common multiple of the integers in the infinite set

$$\{2^9 - 2,\ 3^9 - 3,\ 4^9 - 4,\ \ldots,\ n^9 - n,\ \ldots\}.$$

Solution by Sahib Singh, Clarion College, Clarion, PA.

The least common multiple is infinite because every positive integer n is to be its factor. If we want the greatest common divisor of the members of the set, we note that

$$n^9 - n = n(n - 1)(n + 1)(n^2 + 1)(n^4 + 1) = (n^5 - n)(n^4 + 1).$$

Since $n(n - 1)(n + 1) \equiv 0 \pmod 6$ and $n^5 - n \equiv 0 \pmod 5$, we conclude that

$$n^9 - n \equiv 0 \pmod{30} \text{ for } n = 2, 3, \ldots .$$

By examining the first two terms of the set, we see that the greatest common divisor is 30.

Also solved by Paul. S. Bruckman, Lawrence Somer, and the proposer.

Counting Equilateral Triangles

B-413 *Proposed by Herta T. Freitag, Roanoke, VA.*

For every positive integer n, let U_n consist of the points $j + ke^{2\pi i/3}$ in the Argand plane with

$$j \in \{0, 1, 2, \ldots, n\} \quad \text{and} \quad k \in \{0, 1, \ldots, j\}.$$

Let $T(n)$ be the number of equilateral triangles whose vertices are subsets of U_n. For example, $T(1) = 1$, $T(2) = 5$, and $T(3) = 13$.

(a) Obtain a formula for $T(n)$.
(b) Find all n for which $T(n)$ is an integral multiple of $2n + 1$.

Solution by W. O. J. Moser, McGill University, Montreal, P.Q., Canada.

For the problem as given, $T(3)$ is 15 and not 13 as stated in the problem. The difference may be accounted for by the triangles $\{[2, 2], [1, 0], [3, 1]\}$ and $\{[1, 1], [2, 0], [3, 2]\}$, where $[j, k]$ denotes $j + ke^{2\pi i/3}$. The proposer probably meant to count only the triangles with a side parallel to the real axis. The intended problem is the same as Problem 889, *Math. Mag.* 47 (1974), solution *ibid.* 47 (1974):289-91, where other references are given.

Using the following well-known result one can count various sets of vertices forming equilateral triangles in U_n:

<u>*Lemma*</u>: Let m and r be integers, $m \geq 0$, $r \geq 1$. The number of ordered r-tuples $(a_1, \ldots, a_r)$ of nonnegative integers a_i satisfying $a_1 + \cdots + a_r = m$ is

$$\binom{m + r - 1}{r - 1}.$$

Triples of the form

$$\{[j, k], [j - i, k], [j, k + i]\}, \{[j, k], [j, k - i], [j + i, k]\},$$
$$\{[j + i, k], [j, k + i], [j - i, k - i]\},$$
$$\{[j - i, k], [j, k - i], [j + i, k + i]\}$$

all form equilateral triangles.

Let $A_s(n)$ for $s = 1, 2, 3, 4$ denote the numbers of triples in U of these forms in the order listed. Geometrically, one sees easily that $A_3(n) = A_4(n)$.

Since $[j, k] \in U_n$ if and only if j and k are integers with $0 \le k \le j \le n$, $A_1(n)$ is the number of ordered triples (i, j, k) of nonnegative integers satisfying $1 \le i$ and $k + i \le j \le n$. Letting $x = i - 1$, $y = k$, $z = j - i - k$, and $w = n - j$, we see that $A_1(n)$ is the set of ordered quadruples (x, y, z, w) of nonnegative integers with $x + y + z + w = n - 1$; hence, $A_1(n) = \binom{n+2}{3}$ by the lemma. Other types of triangles may be enumerated similarly.

The answer for the intended problem is

$$T(n) = [n(2n + 1)(n + 2) - \theta_n]/8,$$

with $\theta_n = 0$ for n even and $\theta_n = 1$ for n odd. Hence, $(2n + 1) | T(n)$ iff n is even.

Also solved by Paul S. Bruckman and the proposer.

B-414 *Proposed by Herta T. Freitag, Roanoke, VA.*

Let

$$S_n = L_{n+5} + \binom{n}{2} L_{n+2} - \sum_{i=2}^{n} \binom{i}{2} L_i - 11.$$

Determine all n in $\{2, 3, 4, \ldots\}$ for which S_n is (a) prime; (b) odd.

Solution by Paul S. Bruckman, Concord, CA.

Note that

$$\begin{aligned}\Delta S_n &= S_{n+1} - S_n = L_{n+4} + \binom{n+1}{2} L_{n+3} - \binom{n}{2} L_{n+2} - \binom{n+1}{2} L_{n+1} \\ &= L_{n+4} + \left\{\binom{n+1}{2} - \binom{n}{2}\right\} L_{n+2} = L_{n+4} + nL_{n+3} \\ &= (n + 1)L_{n+4} - nL_{n+3}.\end{aligned}$$

Hence, $S_n = nL_{n+3} + c$, for some constant c. Now

$$S_2 = L_7 + L_4 - L_2 - 11 = 29 + 7 - 3 - 11 = 22;$$

but also,

$$S_2 = 2L_5 + c = 2 \cdot 11 + c = 22 + c.$$

Hence, $c = 0$. Therefore,

(1) $$S_n = nL_{n+3},\ n = 2, 3, 4, \ldots .$$

Clearly, since n and L_{n+3} are each integers greater than 1 (for $n \ge 2$), S_n is *never* prime. In order for S_n to be odd, both n and L_{n+3} must be odd. Now L_n is even iff $3|n$, as is readily seen by inspection of the first few values (mod 2) of the Lucas sequence. Hence, L_{n+3} is odd iff $3 \nmid n$. It follows that S_n is odd iff $n \equiv \pm 1 \pmod 6$.

Also solved by Bob Prielipp, Sahib Singh, and the proposer.

PROPOSALS TABLED

No solutions to problem B-415 were received. The problem was restated by the Elementary Problems Editor in a form not equivalent to the original problem.

No solutions to problem B-416 were received.

Not a Bracket Function

B-417 *Proposed by R. M. Grassl and P. L. Mana, University of New Mexico, Albuquerque, NM*

Here let $[x]$ be the greatest integer in x. Also, let $f(n)$ be defined by

$$f(0) = 1 = f(1),\ f(2) = 2,\ f(3) = 3,$$

and

$$f(n) = f(n-4) + [1 + (n/2) + (n^2/12)] \text{ for } n \in \{4, 5, 6, \ldots\}.$$

Do there exist rational numbers a, b, c, and d such that

$$f(n) = [a + bn + cn^2 + dn^3]?$$

Solution by Paul S. Bruckman, Concord, CA.

We first prove the following:

(1) $$f(12n) = 12n^3 + 15n^2 + 6n + 1,\ n = 0, 1, 2, \ldots .$$

Let S denote the set of all nonnegative integers n for which (1) is true. Since $f(0) = 1$, it is clear that $0 \in S$. Now $f(12n + 12) - f(12n)$

$$= \sum_{k=0}^{2}\Big(f(12n + 4k + 4) - f(12n + 4k)\Big) = \sum_{k=0}^{2}\left(1 + 6n + 2k + 2 + \left[\tfrac{16}{12}(3n+k+1)^2\right]\right)$$

$$= \sum_{k=0}^{2}\left(3 + 6n + 2k + \left[\tfrac{4}{3}\{9n^2 + 6n(k+1) + (k+1)^2\}\right]\right)$$

$$= \sum_{k=0}^{2}\Big\{\Big(3 + 6n + 2k + 12n^2 + 8n(k+1)\Big)\Big\} + [4/3] + [16/3] + 12$$

$$= 1 + 5 + 12 + \sum_{k=0}^{2}\left\{3 + 14n + 12n^2 + (8n+2)k\right\}$$

$$= 3(12n^2 + 14n + 3) + 3(8n + 2) + 18, \text{ or}$$

(2) $$f\Big(12(n+1)\Big) - f(12n) = 36n^2 + 66n + 33.$$

Suppose $n \in S$. Then

$$\begin{aligned} f\Big(12(n+1)\Big) &= 12n^3 + 15n^2 + 6n + 1 + 36n^2 + 66n + 33 \\ &= 12n^3 + 51n^2 + 72n + 34 \\ &= 12(n+1)^3 + 15(n+1)^2 + 6(n+1) + 1. \end{aligned}$$

Hence, $n \in S \Rightarrow (n+1) \in S$. By induction, (1) is proved.

Now, suppose that for *all* $n \geq 0$,

(3) $$f(n) = [a + bn + cn^2 + dn^3]$$

for some rational a, b, c, and d independent of n. Then

$$f(n) = a + bn + cn^2 + dn^3 + e_n,$$

where $e_n = 0(n)$ as $n \to \infty$. In particular, substituting $12n$ for n:

(4) $$f(12n) = a + 12bn + 144cn^2 + 1728dn^3 + e_{12n}.$$

By comparison of (1) and (4), it follows that $12b = 6$, $144c = 15$, $1728d = 12$, i.e.,

(5) $$b = 1/2 = 72/144,\ c = 15/144,\ d = 1/144.$$

Hence,

$$f(n) = \left[\frac{n^3 + 15n^2 + 72n}{144} + a\right],\ n = 0, 1, 2, \ldots . \tag{6}$$

Note that

$$f(5) = f(1) + [1 + 5/2 + 25/12] = 1 + 1 + [55/12] = 6,$$

and

$$f(9) = f(5) + [1 + 9/2 + 81/12] = 6 + 1 + [45/4] = 18.$$

Setting $n = 0$ in (6) yields:

$$f(0) = 1 = [a];$$

however, setting $n = 9$ in (6) yields:

$$f(9) = 18 = [18 + a],$$

which implies $[a] = 0$. This contradiction establishes that the supposition in (3) is false.

Also solved by the proposers.

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