$$S(2,\ 1) = A_{k+3}(1) - 3A_{k+2}(1) + 3A_{k+1}(1) - A_k(1)$$
$$= (k+4) - 3(k+2) + 3(k+1) - k = 1,$$
$$S(3,\ 1) = A_{k+2}(1) - 3A_{k+1}(1) + 3A_k(1) - A_{k-1}(1)$$
$$= (k+2) - 3(k+1) + 3(k) - (k-1) = 0,$$
$$S(j,\ 1) = A_{k+5-j} - 3(1)A_{k+4-j}(1) + 3A_{k+3-j}(1) = A_{k+2-j}(1)$$
$$= (k+5-j) - (k+4-j) + (k+3-j) - (k+2-j)$$
$$= 0 \text{ for } 4 \le j \le k+1.$$

Finally,

$$S(1,\ n) = A_{k+4}(n) - 3A_{k+3}(n) + 3A_{k+2}(n) - A_{k+1}(n),$$
$$S(2,\ n) = A_{k+3}(n) - 3A_{k+2}(n) + 3A_{k+1}(n) - A_k(n) - 1,$$
$$S(j,\ n) = A_{k+5-j}(n) - 3A_{k+4-j}(n) + 3A_{k+3-j}(n) - A_{k+2-j},$$
$$\text{for } 3 \le j \le k+1.$$

REFERENCE

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A NOTE ON TAKE-AWAY GAMES*

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1. SUMMARY

Schwenk [1] considers take-away games where the players alternately remove a positive number of counters from a single pile, the player removing the last counter being the winner. On his initial move, the player moving first can remove at most a given number m of counters. On each subsequent move, a player can remove at most $f(n)$ counters, where n is the number of counters removed by his opponent on the preceding move. In [1], Schwenk solves the case when $f(n)$ is nondecreasing and $f(n) \ge n$. This solution is extended to the case when $f(n)$ is nondecreasing and $f(1) \ge 1$.

2. THE WINNING REPRESENTATION

Let $f(n) \ge 1$ be a nondecreasing function defining a take-away game. If a player whose turn it is to move is confronted with a pile of $n \ge 1$ counters, let $L(n)$ be the minimal number of counters he must remove in order to assure a win. Let $L(0) = \infty$. Note that $L(n) \le n$ for $n \ge 1$ and that equality might hold. Note also that removing k counters from a pile of n is a winning strategy if and only if $f(k) < L(n-k)$.

Theorem 2.1: Suppose $f(k) < L(n-k)$; then $k = L(n)$ if and only if $L(k) = k$.

Proof: Suppose that $L(k) < k$. By removing $L(k)$ counters from a pile of counters, a player can then guarantee he will eventually remove the last of the first k counters, and that he will do this by removing $\ell < k$ counters. His opponent will than face a pile of $n - k$ counters and be able to remove at

*Research partially supported by NSF Grant MCS 72-04591.

most $f(\ell) \leq f(k) < L(n-k)$ counters, implying the opponent cannot win. Thus, removing $L(k) < k$ counters is a winning strategy and k can be minimal winning, i.e., $L(n) = k$, only if $L(k) = k$.

Conversely, if $L(k) = k$ and a player removes fewer than k counters, his opponent can eventually remove the last of the first k counters. Since the opponent will do this by removing $\ell < k$ counters, and since

$$f(\ell) \leq f(k) < L(n - k),$$

we see that the opponent can win. Thus, if $L(k) = k$, then k is minimal winning and $L(n) = k$.

The integers H such that $L(H) = H$ form an increasing, possibly finite sequence H_j satisfying the following theorems.

Theorem 2.2: If $N = \sum_{i=1}^{n} H_{j_k}$ and if $f(H_{j_i}) < H_{j_{i+1}}$ for $i \leq n - 1$, then

$$L(N) = H_{j_i}.$$

Proof: The theorem is true by definition when $n = 1$. Suppose the theorem is true for n and

$$N = \sum_{i=1}^{n+1} H_{j_i},\ f(H_{j_i}) < H_{j_{i+1}},\ i \leq n.$$

Then $f(H_{j_1}) < H_{j_2} = L(N - H_{j_1})$. But since $L(H_{j_1}) = H_{j_1}$, Theorem 2.1 gives

$$L(N) = H_{j_1},$$

completing the proof.

Theorem 2.3: Any positive integer N can be written uniquely as

$$N = \sum_{i=1}^{n} H_{j_i},\ f(H_{j_i}) < H_{j_{i+1}},\ i \leq n - 1.$$

Proof: Let $H_{j_1} = L(N)$ and define

$$H_{j_i} = L\left(N - \sum_{k=1}^{i-1} H_{j_k}\right) \text{ until } \sum_{i=1}^{n} H_{j_i} = N.$$

Then

$$f(H_{j_i}) < L\left(N - \sum_{k=1}^{i-1} H_{j_k} - H_{j_i}\right) = L\left(N - \sum_{k=1}^{i} H_{j_k}\right) = H_{j_{i+1}} \text{ for } i \leq n - 1.$$

Uniqueness follows easily from Theorem 2.2 and a simple induction.

The winning strategy for the game is now clear. Represent the number of counters N as

$$N = \sum_{i=1}^{n} H_{j_i} \text{ with } f(H_{j_i}) < H_{j_{i+1}} \text{ for } i \leq n - 1,$$

and remove H_{j_i} counters.

3. CALCULATION OF THE H_i's

To complete the picture, we have the following theorem on the calculation of the H_i's.

Theorem 3.1: $H_1 = 1$ and if $f(H_j) \geq H_j$, then $H_{j+1} = H_j + H_\ell$, where

$$H_\ell = \min_{i \leq j}\{H_i \mid f(H_i) \geq H_j\}.$$

If $f(H_j) < H_j$, the sequence H_i is finite and H_j is the final term.

Proof: $H_1 = 1$ is obvious. If $f(H_j) \geq H_j$, define $H_j + H_\ell$ as in the statement of the theorem. We must show that $L(H_j + H_\ell) = H_j + H_\ell$, and to do this, we show that

$$f(k) \geq L(H_j + H_\ell - k) \text{ for } 1 \leq k < H_j + H_\ell.$$

First, if $H_\ell < k < H_j + H_\ell$, then $k - H_\ell < H_j$, and so

$$f(k) \geq f(k - H_\ell) \geq L(H_j - (k - H_\ell)) = L(H_j + H_\ell - k).$$

If $k = H_\ell$, then

$$f(k) = f(H_\ell) \geq H_j = L(H_j) = L(H_j + H_\ell - k).$$

If $1 \leq k < H_\ell$, then $f(k) \geq L(H_\ell - k)$. But

$$H_\ell - k = \sum_{i=1}^{n} H_{j_i} \text{ with } f(H_{j_i}) < H_{j_{i+1}} \text{ for } i \leq n - 1, \text{ and } L(H_\ell - k) = H_{j_1}.$$

As a result,

$$H_j + H_\ell - k = \sum_{i=1}^{n} H_{j_i} + H_j,$$

and since H_ℓ is the smallest H_i with $f(H_i) \geq H_j$, it follows that $f(H_{j_n}) < H_j$. Therefore, Theorem 2.2 gives $L(H_j + H_\ell - k) = H_{j_1} = L(H_\ell - k) \leq f(k)$.

We have just shown that $L(H_j + H_\ell) = H_j + H_\ell$. To show that $H_j + H_\ell$ is indeed the next term in the H_i sequence, we need only show that

$$L(H_j + k) < H_j + k \text{ for } 1 \leq k < H_\ell.$$

But such a k can be represented as

$$k = \sum_{i=1}^{n} H_{j_i} \text{ with } f(H_{j_i}) < H_{j_{i+1}} \text{ for } i \leq n - 1,$$

and since $H_{j_n} < H_\ell$, we have $f(H_{j_n}) < H_j$. Hence,

$$L(H_j + k) = L\left(\sum_{i=1}^{n} H_{j_i} + H_j\right) = H_{j_1} < H_j + k$$

by Theorem 2.2, and we have shown that $H_{j+1} = H_j + H_\ell$.

Suppose now that $f(H_j) < H_j$, any positive integer N can be written as $N = k + mH_j$ where $k, m \geq 0$ are integers and $0 < k \leq H_j$. But we can represent k as

$$k = \sum_{i=1}^{n} H_{j_i} \text{ where } f(H_{j_i}) < H_{j_{i+1}} \text{ for } i \leq n - 1,$$

and since $f(H_{j_n}) \leq f(H_j) < H_j$, the representation

$$N = \sum_{i=1}^{n} H_{j_i} + H_j + H_j + \cdots + H_j$$

and Theorem 2.2 tells us that $L(N) = H_{j_1} < H_j$. Thus, H_j is the largest H_i and the theorem is proved.

It may be noted that take-away games with the last player losing may be played with the same strategy but regarding the pile as having one less counter than is actually the case.

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ASSOCIATED STIRLING NUMBERS

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1. INTRODUCTION

For $r \geq 0$, define the integers $s_r(n, k)$ and $S_r(n, k)$ by means of

$$\text{(1.1)} \qquad \left(\log(1-x)^{-1} - \sum_{i=1}^{r} x^i/i\right)^k = \left(\sum_{j=r+1}^{\infty} x^j/j\right)^k = k! \sum_{n=(r+1)k}^{\infty} s_r(n, k)x^n/n!,$$

$$\text{(1.2)} \qquad \left(e^x - \sum_{i=0}^{r} x^i/i!\right)^k = \left(\sum_{j=r+1}^{\infty} x^j/j!\right)^k = k! \sum_{n=(r+1)k}^{\infty} S_r(n, k)x^n/n!.$$

We will call $s_r(n, k)$ the r-associated Stirling number of the first kind, and $S_r(n, k)$ the r-associated Stirling number of the second kind. The terminology and notation are suggested by Comtet [6, pp. 221, 257]. When $r = 0$, we have $s_0(n, k) = (-1)^{n+k}s(n, k)$, where $s(n, k)$ is the Stirling number of the first kind, and $S_0(n, k) = S(n, k)$ is the Stirling number of the second kind. (In Comtet's notation this is true when $r = 1$.) If we define the polynomials $s_{r,n}(y)$ and $S_{r,n}(y)$ by means of

$$\text{(1.3)} \qquad \exp\left(y \sum_{j=r+1}^{\infty} x^j/j\right) = \sum_{n=0}^{\infty} s_{r,n}(y)x^n/n!,$$

$$\text{(1.4)} \qquad \exp\left(y \sum_{j=r+1}^{\infty} x^j/j!\right) = \sum_{n=0}^{\infty} S_{r,n}(y)x^n/n!,$$

it follows immediately that

$$\text{(1.5)} \qquad s_{r,n}(y) = \sum_{j=0}^{[n/r+1]} s_r(n, j)y^j,$$

and

$$\text{(1.6)} \qquad S_{r,n}(y) = \sum_{j=0}^{[n/r+1]} S_r(n, j)y^j.$$

Since the r-associated Stirling numbers of the second kind have appeared in two recent papers [7] and [9], it may be of interest to examine their combinatorial significance, their history, and their basic properties. We do this in §2, §3, and §4 for both the numbers of the first and second kind. Another purpose of this paper is to show how all the results of two recently published articles concerned with Stirling and Bell numbers, [7] and [16] can be generalized by the use of (1.2), (1.4), and (1.6). This is done in §5 and §6. To the writer's knowledge, the r-associated Stirling numbers of the first kind