give a more interesting example, consider the geometric progression $\{a^k\}$ which constitutes a Benford sequence in base b if and only if $a \neq b^{p/q}$ (p, q integers). Setting $a = 3$ and $b' = 9$, we obtain a subset of the positive integers which is not a Benford sequence. Moreover, $Pr(j < 4)_9 = 1$ for the geometric progression $\{3^k\}$. Since $\{3^k\}$ is a Benford sequence in base $b = 8$, we may apply Lemma 1 with $a = 2$, $m = 2$, $n = 3$ to yield $Pr(j < 4)_8 = 2/3$. A comparison of the above probabilities for $b = 8$ and $b' = 9$ shows that the monotonicity statement is false for this example.

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A NEW TYPE MAGIC LATIN 3-CUBE OF ORDER TEN

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A Latin 3-cube of order n is an $n \times n \times n$ cube (n rows, n columns, and n files) in which the numbers $0, 1, 2, \ldots, n-1$ are entered so that each number occurs exactly once in each row, column, and file. A magic Latin 3-cube of order n is an arrangement of n^3 integers in three orthogonal Latin 3-cubes, each of order n (where every ordered triple 000, 001, ..., $n-1$, $n-1$, $n-1$ occurs) such that the sum of the entries in every row, every column, and every file, in each of the four major diagonals (diameters) and in each of the n^2 broken major diagonals is the same; namely, $\frac{1}{2}n(n^3+1)$. We shall list the cubes in terms of n squares of order n that form its different levels from the top square 0 down through (inclusively) square 1, square 2, ..., square $n-1$. We define a broken major diagonal as a path (route) which begins in square 0 and goes through the n different levels (square 0, square 1, ..., square $n-1$) of the cube and passes through precisely one cell in each of the n squares in such a way that no two cells the broken major diagonal traverses are ever in the same file.

The sum of the entries in the n cells that make up a broken major diagonal equals $\frac{1}{2}n(n^3+1)$. A complete system consists of n^2 broken major diagonals, where each broken major diagonal emanates from a cell in square 0, and thus the n^2 broken major diagonals traverse each of the n^3 cells of the cube in n^2 distinct routes. The cube is initially constructed as a Latin 3-cube in which the numbers are expressed in the scale of n $(0, 1, 2, \ldots, n-1)$. However, after adding 1 throughout and converting the numbers to base 10, we have the n^3 numbers $1, 2, \ldots, n^3$ where the sum of the entries in every row, every column, and every file in each of the four major diagonals, and in each of the n^2 broken major diagonals is the same; namely, $\frac{1}{2}n(n^3+1)$.

In this paper, for the first time in mathematics, we construct a magic Latin 3-cube of order ten. In this case, the sum of the numbers in every row, every column, and every file in each of the four major diagonals, and in each of the 10^2 broken major diagonals is the same; namely, $\frac{1}{2}(10)(10^3 + 1) = 5005$.

In Chart 1 we list (by columns) the coordinates of the cells through which 10 broken major diagonals pass. It should be noted that the first digit of the coordinates denotes the row, the second digit the column, and at the right side of each row is the square number in which each cell is to be found. Each one of the 10 broken major diagonals is found under one of the 10 columns, that is, in Chart 1 we find listed by columns 10 broken major diagonals, where each column denotes one broken major diagonal. For example, under column 0, we find the coordinates 00, 99, 55, 66, 11, 88, 77, 22, 44, and 33. These cells determine one broken major diagonal. After adding 1 to each number found in the corresponding 10 cells in the 10 squares, we get

$$764 + 373 + 791 + 588 + 707 + 026 + 445 + 340 + 612 + 359 = \tfrac{1}{2}(10)(10^3 + 1) = 5005.$$

Now, in order to find the remaining 90 broken major diagonals that emanate from square 0, we must construct nine more charts to get Chart 1, Chart 2, ..., Chart X. We need only show (as an example) how to construct Chart 2 from Chart 1 and the Key Chart, since the remaining eight charts (Chart 2, ..., Chart X) are constructed in exactly the same way.

In the Key Chart under column I are the numbers in the same order that are found in Chart 1 under column 0.

In Chart 1, we define the rows as follows:

a (00) row =	00	16	29	35	42	53	64	71	87	98	square 0
a (99) row =	99	41	62	56	84	75	23	10	08	37	square 1
.....											
a (33) row =	33	28	45	04	79	86	90	57	61	12	square 9

Thus, in the Key Chart we have under column I a (00) row, a (99) row, ..., and a (33) row, which is, of course, a restatement in a shorter form of the entire Chart 1.

Now, in the Key Chart, each number under column II which is identical to a number under column I (in the Key Chart) represents the identical row found in Chart 1. Therefore, Chart 2 is written as:

(11) row:	11	89	76	27	58	94	32	03	40	65	square 0
(33) row:	33	28	45	04	79	86	90	57	61	12	square 1
.....											
(99) row:	99	41	62	56	84	75	23	10	08	37	square 9

Then the columns of Chart 2 give 10 more broken major diagonals.

We can find the remaining eight charts—Chart 3, Chart 4, ..., Chart X—in exactly the same way as Chart 2, using the Key Chart in conjunction with Chart 1. (The charts are presented on the following pages.)

It should be noted here that Chart 1 is constructed by superposing two orthogonal Latin squares of order ten. Now, since it is impossible to superpose two Latin squares of order n when $n = 2$ or 6, we may state that this type of magic Latin 3-cube is impossible for order 2 and for order 6.

In the near future, we shall present a more comprehensive general paper in which we consider the general order $4m + 2$ and the powers of prime numbers.

CHART 1

0	1	2	3	4	5	6	7	8	9	
00	16	29	35	42	53	64	71	87	98	square 0
99	41	62	56	84	75	23	10	08	37	square 1
55	97	38	19	60	02	81	24	73	46	square 2
66	50	93	82	07	21	15	48	39	74	square 3
11	89	76	27	58	94	32	03	40	65	square 4
88	72	01	43	95	67	59	36	14	20	square 5
77	34	80	68	13	49	06	92	25	51	square 6
22	05	54	70	31	18	47	69	96	83	square 7
44	63	17	91	26	30	78	85	52	09	square 8
33	28	45	04	79	86	90	57	61	12	square 9

KEY CHART FOR 100 BROKEN MAJOR DIAGONALS

I	II	III	IV	V	VI	VII	VIII	IX	X	
00	11	22	33	44	55	66	77	88	99	square 0
99	33	88	44	22	00	77	55	11	66	square 1
55	22	33	00	77	11	88	99	66	44	square 2
66	44	77	55	00	88	99	11	33	22	square 3
11	00	99	77	66	44	33	22	55	88	square 4
88	55	11	66	99	33	22	44	00	77	square 5
77	88	66	99	11	22	55	00	44	33	square 6
22	66	44	88	55	99	00	33	77	11	square 7
44	77	00	22	33	66	11	88	99	55	square 8
33	99	55	11	88	77	44	66	22	00	square 9

MAGIC LATIN 3-CUBE OF ORDER TEN

Square Number 0

	0	1	2	3	4	5	6	7	8	9
0	763	886	540	979	015	428	601	354	232	197
1	279	963	097	654	832	301	728	186	440	515
2	897	340	463	201	579	632	154	915	028	786
3	140	454	901	063	628	715	879	297	586	332
4	932	228	754	815	163	086	597	401	379	640
5	328	697	132	740	486	563	215	079	954	801
6	554	032	286	128	701	997	363	840	615	479
7	415	779	828	532	397	240	986	663	101	054
8	686	501	315	497	254	179	040	732	863	928
9	001	115	679	386	940	854	432	528	797	263

Square Number 1

	0	1	2	3	4	5	6	7	8	9
0	472	138	264	085	793	616	947	821	359	500
1	385	072	700	921	159	847	416	538	664	293
2	100	864	672	347	285	959	521	093	716	438
3	564	621	047	772	916	493	185	300	238	859
4	059	316	421	193	572	738	200	647	885	964
5	816	900	559	464	638	272	393	785	021	147
6	221	759	338	516	447	000	872	164	993	685
7	693	485	116	259	800	364	038	972	547	721
8	938	247	893	600	321	585	764	459	172	016
9	747	593	985	838	064	121	659	216	400	372

Square Number 2

	0	1	2	3	4	5	6	7	8	9
0	190	924	771	313	808	565	289	637	446	052
1	413	390	852	237	946	689	165	024	571	708
2	952	671	590	489	713	246	037	308	865	124
3	071	537	389	890	265	108	913	452	724	646
4	346	465	137	908	090	824	752	589	613	271
5	665	252	046	171	524	790	408	813	337	989
6	737	846	424	065	189	352	690	971	208	513
7	508	113	965	746	652	471	324	290	089	837
8	224	789	608	552	437	013	781	146	990	365
9	889	008	213	624	371	937	546	765	152	490

Square Number 3

	0	1	2	3	4	5	6	7	8	9
0	987	250	823	431	649	002	794	575	118	366
1	131	487	666	775	218	594	902	350	023	849
2	266	523	087	194	831	718	375	449	602	950
3	323	075	494	687	702	949	231	166	850	518
4	418	102	975	249	387	650	866	094	531	723
5	502	766	318	923	050	887	149	631	475	294
6	875	618	150	302	994	466	587	223	749	031
7	049	931	202	818	566	123	450	787	394	675
8	750	894	549	066	175	331	623	918	287	402
9	694	349	731	550	423	275	018	802	966	187

Square Number 4

	0	1	2	3	4	5	6	7	8	9
0	606	541	355	727	434	999	010	162	883	278
1	827	706	478	062	583	110	699	241	955	334
2	578	115	906	810	327	083	262	734	499	641
3	255	962	710	406	099	634	527	878	341	183
4	783	899	662	534	206	441	378	910	127	055
5	199	078	283	655	941	306	834	427	762	510
6	362	483	841	299	610	778	106	555	034	927
7	934	627	599	383	178	855	741	006	210	462
8	041	310	134	978	862	227	455	683	506	799
9	410	234	027	141	755	562	983	399	678	806

Square Number 5

	0	1	2	3	4	5	6	7	8	9
0	525	069	488	842	157	230	376	903	691	714
1	642	825	114	303	091	976	530	769	288	457
2	014	988	225	676	442	391	703	857	130	569
3	788	203	786	125	330	557	042	614	469	991
4	891	630	503	057	725	169	414	276	942	388
5	930	314	791	588	269	425	657	142	803	076
6	403	191	669	730	576	814	925	088	357	242
7	257	542	030	491	914	688	869	325	776	103
8	369	476	957	214	603	742	188	591	025	830
9	176	757	342	969	888	003	291	430	514	625

Square Number 6

	0	1	2	3	4	5	6	7	8	9
0	044	312	136	698	961	777	453	280	505	829
1	598	644	929	480	305	253	077	812	736	161
2	329	236	744	553	198	405	880	661	977	012
3	836	780	653	944	477	061	398	529	112	205
4	605	577	080	361	844	912	129	753	298	436
5	277	429	805	036	712	144	561	998	680	353
6	180	905	512	877	053	629	244	336	461	798
7	761	098	377	105	229	536	612	444	853	980
8	412	153	261	729	580	989	936	005	344	677
9	953	861	498	212	636	380	705	177	029	544

Square Number 7

	0	1	2	3	4	5	6	7	8	9
0	239	773	617	104	582	351	868	096	920	445
1	904	139	545	986	720	068	251	473	317	682
2	745	017	339	968	604	820	496	182	551	273
3	417	396	168	539	851	282	704	945	673	020
4	120	951	296	782	439	573	645	368	004	817
5	051	845	420	217	373	639	982	504	196	768
6	696	520	973	451	268	145	039	717	882	304
7	382	204	751	620	045	917	173	839	468	596
8	873	668	082	345	996	404	517	220	739	151
9	568	482	804	073	117	796	320	651	245	939

Square Number 8

	0	1	2	3	4	5	6	7	8	9
0	311	495	909	556	270	884	122	748	067	633
1	056	511	233	148	467	722	384	695	809	970
2	433	709	811	022	956	167	648	570	284	395
3	609	848	522	211	184	370	456	033	995	767
4	567	084	348	470	611	295	933	822	756	109
5	784	133	667	309	895	911	070	256	548	422
6	948	267	095	684	322	533	711	409	170	856
7	870	356	484	967	733	009	595	111	622	248
8	195	922	770	833	048	656	209	367	411	584
9	222	670	156	795	509	448	867	984	333	011

Square Number 9

	0	1	2	3	4	5	6	7	8	9
0	858	607	092	260	326	143	535	419	774	981
1	760	258	381	519	674	435	843	907	192	026
2	681	492	158	735	060	574	919	226	343	807
3	992	119	235	358	543	826	660	781	007	474
4	274	743	819	626	958	307	081	135	460	592
5	443	581	974	892	107	058	726	360	219	635
6	019	374	707	943	835	281	458	692	526	160
7	126	860	643	074	481	792	207	558	935	319
8	507	035	426	181	719	960	392	874	658	243
9	335	926	560	407	292	619	174	043	881	758

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COMPLEX FIBONACCI NUMBERS

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1. INTRODUCTION

In this note, a new approach is taken toward the significant extension of Fibonacci numbers into the complex plane. Two differing methods for defining such numbers have been considered previously by Horadam [4] and Berzsenyi [2]. It will be seen that the new numbers include Horadam's as a special case, and that they have a symmetry condition which is not satisfied by the numbers considered by Berzsenyi.

The latter defined a set of complex numbers at the Gaussian integers, such that the characteristic Fibonacci recurrence relation is satisfied at any horizontal triple of adjacent points. The numbers to be defined here will have the symmetric condition that the Fibonacci recurrence occurs on any horizontal or vertical triple of adjacent points.

Certain recurrence equations satisfied by the new numbers are outlined, and using them, some interesting new Fibonacci identities are readily obtained. Finally, it is shown that the numbers generalize in a natural manner to higher dimensions.

2. THE COMPLEX FIBONACCI NUMBERS

The numbers, to be denoted by $G(n, m)$, will be defined at the set of Gaussian integers $(n, m) = n + im$, where $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$. By direct analogy with the classical Fibonacci recurrence

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1, \tag{2.1}$$

the numbers $G(n, m)$ will be required to satisfy the following two-dimensional recurrence

$$G(n + 2, m) = G(n + 1, m) + G(n, m), \tag{2.2}$$

$$G(n, m + 2) = G(n, m + 1) + G(n, m), \tag{2.3}$$

where $$G(0, 0) = 0,\ G(1, 0) = 1,\ G(0, 1) = i,\ G(1, 1) = 1 + i. \tag{2.4}$$

The conditions (2.2), (2.3), and (2.4) are sufficient to specify the unique value of $G(n, m)$ at each point (n, m) in the plane, and the actual value of $G(n, m)$ will now be obtained.

From (2.2), the case $m = 0$ gives

$$G(n + 2, 0) = G(n + 1, 0) + G(n, 0);\ G(0, 0) = 0,\ G(1, 0) = 1$$