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ON RECIPROCAL SERIES RELATED TO FIBONACCI NUMBERS WITH SUBSCRIPTS IN ARITHMETIC PROGRESSION

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1. INTRODUCTION

Recently, interest has been shown in summing infinite series of reciprocals of Fibonacci numbers [1], [2], and [3]. As V. E. Hoggatt, Jr., and Marjorie Bicknell state [2]: "It is not easy, in general, to derive the sum of a series whose terms are reciprocals of Fibonacci numbers such that the subscripts are terms of geometric progressions." It seems even more difficult if the subscripts are in arithmetic progression. To take a very simple example, to my knowledge the series

$$\sum_{1}^{\infty} \frac{1}{F_n} \tag{1.1}$$

has not been evaluated in closed form, although Brother U. Alfred has derived formulas connecting it with other highly convergent series [4].

In this note, we develop formulas for closely related series of the form

$$\sum_{0}^{\infty} \frac{1}{F_{an+b} + c} \tag{1.2}$$

for certain values of a, b, and c. Examples include the following:

$$(1.3)\qquad \sum_0^\infty \frac{1}{F_{2n+1}+1} = \sqrt{5}/2, \quad \sum_0^\infty \frac{1}{F_{2n+1}+2} = 3\sqrt{5}/8,$$
$$\sum_0^\infty \frac{1}{F_{2n+1}+5} = 5\sqrt{5}/22, \quad \sum_0^\infty \frac{1}{F_{2n+1}+13} = 7\sqrt{5}/58.$$

In fact, much more than this is true. Each of these series may be further broken down into a remarkable set of symmetric series illustrated by the following examples:

$$\sum_0^\infty \frac{1}{F_{14n+1}+13} = (\sqrt{5}+2)/58, \quad \sum_0^\infty \frac{1}{F_{14n+13}+13} = (\sqrt{5}-2)/58,$$
$$\sum_0^\infty \frac{1}{F_{14n+3}+13} = (\sqrt{5}+5/3)/58, \quad \sum_0^\infty \frac{1}{F_{14n+11}+13} = (\sqrt{5}-5/3)/58,$$
$$(1.4)\qquad \sum_0^\infty \frac{1}{F_{14n+5}+13} = (\sqrt{5}+1)/58, \quad \sum_0^\infty \frac{1}{F_{14n+9}+13} = (\sqrt{5}-1)/58,$$
$$\sum_0^\infty \frac{1}{F_{14n+7}+13} = \sqrt{5}/58.$$

It will be noted that the sum of the series in (1.4) agrees with that given in (1.3)—namely, $7\sqrt{5}/58$—since the rational terms cancel out in pairs. Also, the reader will have noticed the use of $c = 1$, 2, 5, and 13 in these examples. They are, of course, the Fibonacci numbers with odd subscripts. Unfortunately, the methods of this note do not apply to values of c which are Fibonacci numbers with even subscripts.

2. MAIN RESULTS

The main results of this note are summarized in three theorems: Theorem I provides a formulation of series of the form (1.3); Theorem II gives finer results where the sums are broken down into individual series similar to those in (1.4); Theorem III reveals even more detailed information in the form of explicit formulas for the partial sums of series in Theorem II.

In the following discussion, it will be assumed that K represents an odd integer and that t is an integer in the range $-(K-1)/2$ to $(K-1)/2$ inclusive.

Theorem I:

$$S(K) = \sum_0^\infty \frac{1}{F_{2n+1}+F_K} = K\sqrt{5}/2L_K.$$

Theorem II:

$$S(K,\ t) = \sum_0^\infty \frac{1}{F_{(2n+1)K+2t}+F_K} = (\sqrt{5}-5F_t/L_t)/2L_K \qquad t \text{ even},$$
$$= (\sqrt{5}-L_t/F_t)/2L_K \qquad t \text{ odd}.$$

Theorem III:

$$S_N(K,\ t) = \sum_0^N \frac{1}{F_{(2n+1)K+2t}+F_K} = \left(\frac{L_{(N+1)K+t}}{F_{(N+1)K+t}} - \frac{5F_t}{L_t}\right)/2L_K \qquad N \text{ even},\ t \text{ even} \qquad \text{(a)}$$

(continued)

$$= \left(\frac{5F_{(N+1)K+t}}{L_{(N+1)K+t}} - \frac{L_t}{F_t}\right)/2L_K \qquad N \text{ even, } t \text{ odd;} \tag{b}$$

$$= \left(\frac{5F_{(N+1)K+t}}{L_{(N+1)K+t}} - \frac{5F_t}{L_t}\right)/2L_K \qquad N \text{ odd, } t \text{ even;} \tag{c}$$

$$= \left(\frac{L_{(N+1)K+t}}{F_{(N+1)K+t}} - \frac{L_t}{F_t}\right)/2L_K \qquad N \text{ odd, } t \text{ odd.} \tag{d}$$

3. ELEMENTARY RESULTS

We shall adopt the usual Fibonacci and Lucas number definitions:

$$F_{n+2} = F_{n+1} + F_n \text{ with } F_0 = 0 \text{ and } F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n \text{ with } L_0 = 2 \text{ and } L_1 = 1.$$

We shall also employ the well-known Binet forms:

$$F_n = (\alpha^n - \beta^n)/\sqrt{5} \text{ and } L_n = \alpha^n + \beta^n$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. Other elementary results which will be required include $\alpha\beta = -1$ and $F_{2a} = F_a L_a$.

4. PROOF OF MAIN RESULTS

To prove Theorems I, II, and III, it will be sufficient to prove Theorem III together with several short lemmas that establish the connection with Theorems I and II.

Lemma 1: $\displaystyle\lim_{n\to\infty} \frac{L_n}{F_n} = \lim_{n\to\infty} \frac{5F_n}{L_n} = \sqrt{5}.$

Proof: From the Binet forms, we have

$$\frac{L_n}{F_n} = \frac{\sqrt{5}(\alpha^n + \beta^n)}{(\alpha^n - \beta^n)} = \frac{\sqrt{5}(1 + (-1)^n\beta^{2n})}{(1 - (-1)^n\beta^{2n})} \to \sqrt{5} \text{ as } n \to \infty.$$

The second part follows immediately, since $5/\sqrt{5} = \sqrt{5}$.

Lemma 2: $\displaystyle\frac{L_t}{F_t} = -\frac{L_{-t}}{F_{-t}}.$

Proof: Again using the Binet forms, we have

$$-\frac{L_{-t}}{F_{-t}} = \frac{-\sqrt{5}(\alpha^{-t} + \beta^{-t})}{(\alpha^{-t} - \beta^{-t})} = \frac{-\sqrt{5}((-1)^t\beta^t + (-1)^t\alpha^t)}{((-1)^t\beta^t - (-1)^t\alpha^t)}$$

$$= \frac{-\sqrt{5}(\beta^t + \alpha^t)}{(\beta^t - \alpha^t)} = \frac{\sqrt{5}(\alpha^t + \beta^t)}{(\alpha^t - \beta^t)} = \frac{L_t}{F_t}. \qquad \text{Q.E.D.}$$

Theorem II may therefore be deduced from Theorem III and Lemma 1 and taking the limits as N approaches infinity. Summation of the results of Theorem II over the K values of t ranging from $-(K - 1)/2$ to $(K - 1)/2$ inclusive implies the truth of Theorem I, since the rational terms cancel out in pairs (as guaranteed by Lemma 2).

Before proceeding to the proof of Theorem III, we will need the results of the following four lemmas.

Lemma 3: $F_{a+2b} + F_a = F_{a+b} \cdot L_b$ for b even.

Proof: Since b is even, $(\alpha\beta)^b = +1$.

$$\begin{aligned}\text{RHS} &= (\alpha^{a+b} - \beta^{a+b})(\alpha^b + \beta^b)/\sqrt{5}\\ &= (\alpha^{a+2b} + \alpha^{a+b} \cdot \beta^b - \beta^{a+b} \cdot \alpha^b - \beta^{a+2b})/\sqrt{5}\\ &= (\alpha^{a+2b} - \beta^{a+2b} + (\alpha\beta)^b(\alpha^a - \beta^a))/\sqrt{5}\\ &= F_{a+2b} + F_a = \text{LHS}.\end{aligned}$$

Lemma 4: $F_{2a+b} + F_b = F_a \cdot L_{a+b}$ for a odd.

Proof: Since a is odd, $(\alpha\beta)^a = -1$.

$$\begin{aligned}\text{RHS} &= (\alpha^a - \beta^a)(\alpha^{a+b} + \beta^{a+b})/\sqrt{5}\\ &= (\alpha^{2a+b} + \alpha^a \cdot \beta^{a+b} - \beta^a \cdot \alpha^{a+b} - \beta^{2a+b})/\sqrt{5}\\ &= (\alpha^{2a+b} - \beta^{2a+b} - (\alpha\beta)^a(\alpha^b - \beta^b))/\sqrt{5}\\ &= F_{2a+b} + F_b = \text{LHS}.\end{aligned}$$

Lemma 5: $L_a \cdot L_b - 5F_a \cdot F_b = 2L_{a-b}$ for b even.

Proof: Since b is even, $(\alpha\beta)^b = +1$.

$$\begin{aligned}\text{LHS} &= (\alpha^a + \beta^a)(\alpha^b + \beta^b) - (\alpha^a - \beta^a)(\alpha^b - \beta^b)\\ &= \alpha^{a+b} + \alpha^a \cdot \beta^b + \beta^a \cdot \alpha^b + \beta^{a+b} - \alpha^{a+b} + \alpha^a \cdot \beta^b + \beta^a \cdot \alpha^b - \beta^{a+b}\\ &= (\alpha\beta)^b(\alpha^{a-b} + \beta^{a-b} + \alpha^{a-b} + \beta^{a-b})\\ &= 2(\alpha^{a-b} + \beta^{a-b}) = 2L_{a-b} = \text{RHS}.\end{aligned}$$

Lemma 6: $L_a \cdot F_b - F_a \cdot L_b = 2F_{a-b}$ for b odd.

Proof: Since b is odd, $(\alpha\beta)^b = -1$.

$$\begin{aligned}\text{LHS} &= ((\alpha^a + \beta^a)(\alpha^b - \beta^b) - (\alpha^a - \beta^a)(\alpha^b + \beta^b))/\sqrt{5}\\ &= (\alpha^{a+b} - \alpha^a \cdot \beta^b + \beta^a \cdot \alpha^b - \beta^{a+b} - \alpha^{a+b} - \alpha^a \cdot \beta^b + \beta^a \cdot \alpha^b + \beta^{a+b})/\sqrt{5}\\ &= -(\alpha\beta)^b(\alpha^{a-b} - \beta^{a-b} + \alpha^{a-b} - \beta^{a-b})/\sqrt{5}\\ &= 2(\alpha^{a-b} - \beta^{a-b})/\sqrt{5} = 2F_{a-b} = \text{RHS}.\end{aligned}$$

We shall prove part (a) of Theorem III in full and leave the details of parts (b), (c), and (d) to the reader, since they follow exactly the same pattern. In the discussion that follows, we will assume both N and t to be even.

We shall proceed by induction on N.

$N = 0$: We must prove that

$$\frac{1}{F_{K+2t} + F_K} = \left(\frac{L_{K+t}}{F_{K+t}} - \frac{5F_t}{L_t}\right)/2L_K.$$

Using Lemma 3 with $a = K$ and $b = t$ gives $F_{K+2t} + F_K = F_{K+t} \cdot L_t$. Hence

$$\text{LHS} = \frac{1}{F_{K+t} \cdot L_t} \quad \text{and} \quad \text{RHS} = \frac{L_{K+t} \cdot L_t - 5F_{K+t} \cdot F_t}{2F_{K+t} \cdot L_t \cdot L_K}.$$

Using Lemma 5 with $a = K + t$ and $b = t$ gives $L_{K+t} \cdot L_t - 5F_{K+t} \cdot F_t = 2L_K$. Hence

$$\text{RHS} = \frac{2L_K}{2F_{K+t} \cdot L_t \cdot L_K} = \frac{1}{F_{K+t} \cdot L_t} = \text{LHS}.$$

Assuming that Theorem III (a) is true for $N = M$ (where M is even), we must prove it true for $N = M + 2$. Hence, the sum of the two extra terms on the LHS corresponding to $N = M + 1$ and $N = M + 2$ must equal the difference in the RHS formulas for $N = M + 2$ and $N = M$. Therefore, we must prove that

$$\frac{1}{F_{(2M+3)K+2t} + F_K} + \frac{1}{F_{(2M+5)K+2t} + F_K} = \left(\frac{L_{(M+3)K+t}}{F_{(M+3)K+t}} - \frac{5F_t}{L_t}\right)/2L_K - \left(\frac{L_{(M+1)K+t}}{F_{(M+1)K+t}} - \frac{5F_t}{L_t}\right)/2L_K.$$

To simplify the following algebra, we introduce the odd integer P, where

$$P = (M + 1)K + t.$$

This means that we must now prove that

$$\frac{1}{F_{2P+K} + F_K} + \frac{1}{F_{2P+3K} + F_K} = \left(\frac{L_{P+2K}}{F_{P+2K}} - \frac{L_P}{F_P}\right)/2L_K$$

Using Lemma 4 with $a = P$ and $b = K$ gives

$$F_{2P+K} + F_K = F_P \cdot L_{P+K}.$$

Using Lemma 3 with $a = K$ and $b = P + K$ gives

$$F_{2P+3K} + F_K = F_{P+2K} \cdot L_{P+K};$$

$$\text{LHS} = \frac{1}{F_P \cdot L_{P+K}} + \frac{1}{F_{P+2K} \cdot L_{P+K}} = \frac{F_{P+2K} + F_P}{F_P \cdot L_{P+K} \cdot F_{P+2K}}$$

Using Lemma 4 with $a = K$ and $b = P$ gives

$$F_{P+2K} + F_P = F_K \cdot L_{P+K};$$

$$\text{LHS} = \frac{F_K \cdot L_{P+K}}{F_P \cdot L_{P+K} \cdot F_{P+2K}} = \frac{F_K}{F_P \cdot F_{P+2K}};$$

$$\text{RHS} = \frac{L_{P+2K} \cdot F_P - F_{P+2K} \cdot L_P}{2F_{P+2K} \cdot F_P \cdot L_K}.$$

Using Lemma 6 with $a = P + 2K$ and $b = P$ gives

$$L_{P+2K} \cdot F_P - F_{P+2K} \cdot L_P = 2F_{2K} = 2F_K \cdot L_K;$$

$$\text{RHS} = \frac{2F_K \cdot L_K}{2F_{P+2K} \cdot F_P \cdot L_K} = \frac{F_K}{F_{P+2K} \cdot F_P} = \text{LHS}.$$

5. EXTENSION TO LUCAS NUMBERS

Similar results may be obtained by substituting Lucas numbers for the Fibonacci numbers in (1.2). In this case, however, even subscripts are required. Examples equivalent to those in (1.3) include the following:

(5.1)

$$\sum_0^\infty \frac{1}{L_{2n} + 3} = (2\sqrt{5} + 1)/10 \qquad \sum_0^\infty \frac{1}{L_{2n} + 7} = (4\sqrt{5} + 5/3)/30$$

$$\sum_0^\infty \frac{1}{L_{2n} + 18} = (6\sqrt{5} + 2)/80 \qquad \sum_0^\infty \frac{1}{L_{2n} + 47} = (8\sqrt{5} + 15/7)/210$$

These series may also be broken down into subseries similar to those in (1.4). For example:

$$\sum_0^\infty \frac{1}{L_{12n} + 18} = (\sqrt{5} + 2)/80$$

(5.2)
$$\sum_0^\infty \frac{1}{L_{12n+2} + 18} = (\sqrt{5} + 5/3)/80 \qquad \sum_0^\infty \frac{1}{L_{12n+10} + 18} = (\sqrt{5} - 5/3)/80$$
$$\sum_0^\infty \frac{1}{L_{12n+4} + 18} = (\sqrt{5} + 1)/80 \qquad \sum_0^\infty \frac{1}{L_{12n+8} + 18} = (\sqrt{5} - 1)/80$$

$$\sum_0^\infty \frac{1}{L_{12n+6} + 18} = \sqrt{5}/80$$

Notice that, in this case, the rational terms occur in pairs except for the first series. This explains the presence of the residual rational terms in (5.1) above.

The following three theorems (IV-V) summarize the above results. They are given without proof, since the methods required exactly parallel those of Section 4. In these theorems, We assume that K is an even integer and that t is an integer in the range $-K/2$ to $K/2 - 1$ inclusive.

Theorem IV:

$$T(K) = \sum_0^\infty \frac{1}{L_{2n} + L_K} = K\sqrt{5}/10F_K + 1/2L_{K/2}^2 \qquad K/2 \text{ even,}$$
$$= K\sqrt{5}/10F_K + 1/10F_{K/2}^2 \qquad K/2 \text{ odd.}$$

Theorem V:

$$T(K, t) = \sum_0^\infty \frac{1}{L_{(2n+1)K+2t} + L_K} = (\sqrt{5} - 5F_t/L_t)/10F_K \qquad t \text{ even,}$$
$$= (\sqrt{5} - L_t/F_t)/10F_K \qquad t \text{ odd.}$$

Theorem VI:

$$T_N(K, t) = \sum_0^N \frac{1}{L_{(2n+1)K+2t} + L_K}$$
$$= \left(\frac{5F_{(N+1)K+t}}{L_{(N+1)K+t}} - \frac{5F_t}{L_t}\right)/10F_K \qquad t \text{ even,}$$
$$= \left(\frac{L_{(N+1)K+t}}{F_{(N+1)K+t}} - \frac{L_t}{F_t}\right)/10F_K \qquad t \text{ odd.}$$

6. A TANTALIZING PROBLEM

If we let $K = 0$ in Theorem V or VI, we find that they give divergent series. However, if we formally substitute $K = 0$ into Theorem IV (without, as yet, any mathematical justification), we find that the LHS is finite, namely:

(6.1)
$$\sum_0^\infty \frac{1}{L_{2n} + 2} = .64452\ 17830\ 67274\ 44209\ 92731\ 19038$$

(to 30 decimal places). The RHS, however, contains the indeterminate form K/F_K.

If we take the liberty of defining a Fibonacci function such as

$$\begin{aligned} f(x) &= (\alpha^x - (-1)^x\alpha^{-x})/\sqrt{5} \\ &= (\alpha^x - (\cos \pi x + i \sin \pi x)\alpha^{-x})/\sqrt{5} \\ &= ((\alpha^x - \cos \pi x \cdot \alpha^{-x}) - i \sin \pi x \cdot \alpha^{-x})/\sqrt{5} \end{aligned}$$

and differentiate with respect to x, the real part becomes:

$$\mathrm{Re}[f'(x)] = (\ln \alpha \cdot \alpha^x + \pi \sin \pi x \cdot \alpha^{-x} + \cos \pi x \cdot \ln \alpha \cdot \alpha^{-x})/\sqrt{5}$$

and

$$\mathrm{Re}[f'(0)] = (\ln \alpha \cdot 1 + \pi \cdot 0 \cdot 1 + 1 \cdot \ln \alpha \cdot 1)/\sqrt{5} = 2 \ln \alpha/\sqrt{5}.$$

Substituting this value into the RHS of Theorem IV gives:

$$1/(4 \ln \alpha) + 1/8 = .64452\ 17303\ 08756\ 88440\ 03306\ 51529 \tag{6.2}$$

(to 30 decimal places). The difference between the values in (6.1) and (6.2) is obvious, but can any reader resolve this most tantalizing problem?

7. CONCLUSIONS

In this note, we have established explicit formulas for a number of series of the form

$$\sum_0^\infty \frac{1}{F_{an+b} + c} \quad \text{and} \quad \sum_0^\infty \frac{1}{L_{an+b} + c} \tag{7.1}$$

for certain values of a, b, and c positive. Similar results apply for c negative, but because of the possibility of a zero denominator, the series must begin with the term in which $an + b > K$. This leads to less elegant formulas, such as the following:

$$\sum_0^\infty \frac{1}{F_{6n+5} - 2} = (5 - \sqrt{5})/8$$

$$\sum_0^\infty \frac{1}{F_{6n+7} - 2} = (3 - \sqrt{5})/8 \qquad \sum_0^\infty \frac{1}{F_{6n+9} - 2} = (5/2 - \sqrt{5})/8. \tag{7.2}$$

Summing these three series gives

$$\sum_0^\infty \frac{1}{F_{2n+5} - 2} = (21/2 - 3\sqrt{5})/8, \tag{7.3}$$

where the symmetric form of (1.4) appears to have been lost. Similar results may be obtained using the Lucas numbers in (7.1). We leave the reader to investigate these formulas and to determine the true value of the series:

$$\sum_0^\infty \frac{1}{L_{2n} + 2}. \tag{7.4}$$

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ON THE EQUATION $\sigma(m)\sigma(n) = (m + n)^2$

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1. A pair of positive integers m and n are called amicable if

$$\sigma(m)\sigma(n) = (m + n)^2 \quad \text{and} \quad \sigma(m) = \sigma(n).$$

Although over a thousand pairs of amicable numbers are known, no pairs of relatively prime amicable numbers are known. Some necessary conditions for existence of such numbers are given in [1], [2], and [3].

In this paper, we show that some of the conditions are also necessary for the existence of m and n satisfying

$$\sigma(m)\sigma(n) = (m + n)^2, \tag{1}$$

and

$$(m, n) = 1. \tag{2}$$

In particular we prove

Theorem: If m and n satisfy (1) and (2), mn is divisible by at least twenty-two distinct primes.

Corollary (Hagis [3]): The product of relatively prime amicable numbers are divisible by twenty-two distinct primes.

2. Throughout this paper, let m and n be positive integers satisfying (1) and (2), and let

$$mn = \prod_{i=1}^{r} p_i^{a_i}$$

where $p_1 < \cdots < p_r$ are primes and the a_i's are positive integers. Since σ is multiplicative,

$$\prod_{i=1}^{r} \sigma(p_i^{a_i}) = \sigma(mn) = (m + n)^2.$$

If k and a are positive integers, p is a prime and if $p^a | k$ and $p^{a+1} \nmid k$, then we write $p^a || k$. $\omega(k)$ denotes the number of distinct prime factors of k.

Lemma 1: $\sigma(mn)/mn > 4$.

Proof: By (1) and (2)

$$\frac{\sigma(mn)}{mn} = \frac{(m+n)^2}{mn} = 4 + \frac{(m-n)^2}{mn} > 4. \quad \text{Q.E.D.}$$

Lemma 2: If q is a prime, $q | mn$ and if $p^a || mn$, $q \nmid \sigma(p^a)$.

Proof: Suppose q is a prime, $q | mn$, $p^a || mn$, and $q | \sigma(p^a)$. Since

$$\sigma(p^a) \,|\, (m + n)^2,$$

$q | m + n$. Then $q | m$ and $q | n$, contradicting (2). Q.E.D.